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Analysis of Dependent Variables Following FGM Bivariate Generalized Burr Distribution Based on Economic Data

Hiba Z. Muhammed[1] and Samira R. Abozaid[2, *]

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Keywords

Generalized Burr distribution; compound Weibull distribution; FGM copula; Maximum likelihood estimation.

Abstract

The generalized Burr (GB) distribution is a flexible distribution that is used to describe many types of data. This distribution has a flexible hazard function, which can take a decreasing, approximately constant, or unimodal shape over time. This makes the GB distribution one of the most applicable in many fields. This paper introduces a bivariate GB distribution that was created by using the Farlie-Gumbel-Morgenstern (FGM) copula and the univariate GB. Some mathematical properties of FGM bivariate GB (FGMBGB) distribution are obtained, such as the reversed hazard function, product moment, and moment generating function. The maximum likelihood estimation method is used to estimate the unknown parameters, and a simulation study was conducted to assess the performance of the estimators under different sample sizes. The performance was evaluated using different measures. The simulation results show that all estimation criteria improve significantly with larger sample sizes. Moreover, dependence measures showed better convergence to true values as n increased. The model was also applied to real economic data involving GDP growth and exports of goods and services, where the FGMBGB model showed a good fit according to the Anderson-Darling test and model selection criteria. These results confirm the flexibility and applicability of the FGMBGB model in modeling economic data and studying dependencies in various fields.

Mathematical Subject Classification: 62H05, 62P20, 62H20, 62F12

1. Introduction

The focus of this study is on bivariate generalized Burr distribution; most applications of statistics require more than one random variable model of probabilistic phenomena that take into account a large number of random variables. These models allow us to study the interactions between the

✉ Corresponding author*: samiraramadan760@gmail.com
[1]Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt.
[2]Master's Researcher, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt.

associated components. Sklar (1959) presents a crucial insight into the role of copulas in connecting joint distribution functions with their univariate marginal distributions. Copulas are functions that join multivariate distributions to their univariate marginal distributions, allowing for an understanding of the relationships and dependencies among random variables. Copulas are used to model joint distributions more accurately as they can express complex dependence patterns between variables. Several extensions of the FGM copula have been proposed to improve flexibility in modeling dependence. Sharifonnasabi et al. (2019) constructed a class of bivariate distributions using copula FGM copulas, and examined their properties, Recent advancements, such as the work by Chandra et al. (2024), have introduced a bivariate iterated Farlie–Gumbel–Morgenstern (FGM) model for Rayleigh marginals, providing a comprehensive analysis of its properties and dependence measures. Such developments significantly improve the modeling capacity of copula-based approaches, facilitating more precise reliability assessments in complex systems across various engineering and applied sciences. These findings underscore the ongoing research efforts aimed at extending the applicability of copula functions for dependence modeling, especially in applications requiring stronger dependent structures. Almetwally et al. (2020) introduced bivariate Weibull distribution by using the FGM copula function, and some properties of this distribution are obtained. Muhammad et al. (2021) addressed dependency measures in new bivariate models based on copula functions, exploring the properties of Spearman's rho and Kendall's Tau within various types of copulas. The study involved different families of copulas, including Farlie-Gumbel-Morgenstern, Ali-Mikhail-Haq, Plackett, and Clayton. For more information on the copula and its application in FGM copula see Ebaid et al. (2020), El-Sherpieny et al. (2022), and Susam (2025).
Muhammed (2019) introduced the BGB distribution according to the Marshall and Olkin method but in this paper, we study the bivariate generalized Burr distribution using the Farlie-Gumble-Morgenstern (FGM) family. The FGM family was discussed by Morgenstern (1956), Gumbel (1958), and Farlie (1960), and it is given as follows:

The cumulative FGM copula function is given as follows:

$$C\left(F(x_1),F(x_2)\right) = F(x_1)F(x_2)\{1+\theta[1-F(x_1)][1-F(x_2)]\}, \tag{1}$$

The bivariate density function is given by

$$c(F(x_1),F(x_2)) = 1+\theta\{[1-2F(x_1)][1-2F(x_2)]\}, \tag{2}$$

with copula parameter $-1<\theta<1$.

Dubey (1968) introduced the generalized Burr distribution. It is assumed that the conditional random variable X follows the Weibull distribution, and its scale parameter follows gamma distribution. The generalized Burr (GB) distribution with parameters $\alpha, \vartheta\ and\ \delta$ has the following form for its probability density function (PDF) given by

$$f(x) = \frac{\alpha\,\vartheta}{\delta}\; x^{\alpha-1}(1+\frac{x^{\alpha}}{\delta})^{-\vartheta-1} \qquad ,x\geq 0, \alpha, \vartheta, and\ \delta>0. \tag{3}$$

Hence, the CDF of GB $dist^n$ is given by

$$\mathrm{F(x) = P(X \leq x)} = 1-(1+\frac{\mathrm{x}^{\alpha}}{\delta})^{-\vartheta} \qquad ,x\geq 0, and\ \alpha, \vartheta, \delta>0, \tag{4}$$

where α, ϑ are the shape parameters and δ is the scale parameter, respectively.

The PDF and the CDF for GB distribution when various parameter values are taken into account are shown in Figure 1. The GB distribution decreases if $\alpha \leq 1$ and unimodal if $\alpha > 1$, with the right skewness.

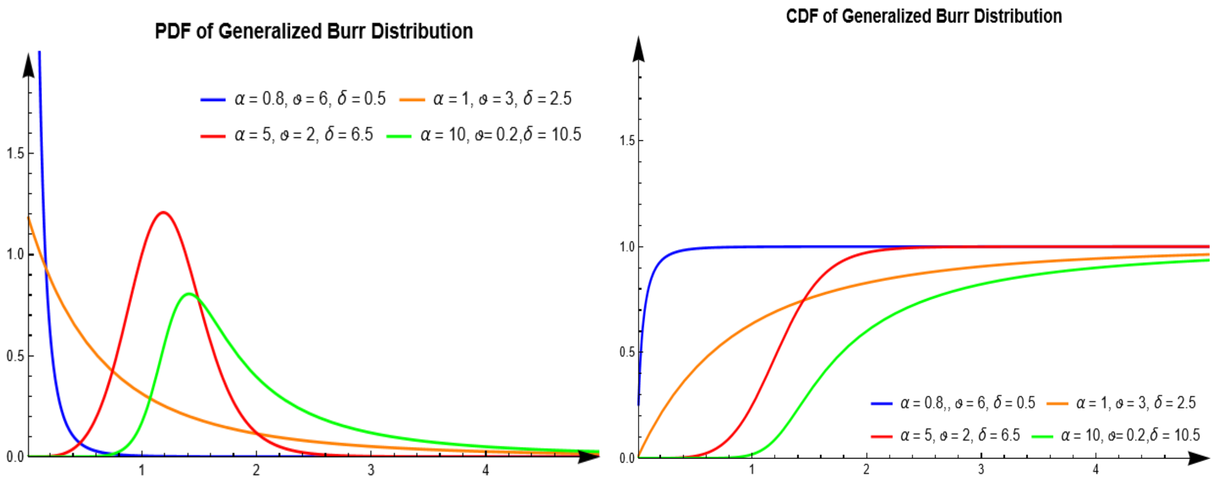


Figure 1. The PDF and the CDF of the GB distribution

The cumulative distribution function of the Burr distribution is given by

$$F(x) = P(X \leq x) = 1 - (1 + x^{\alpha})^{-\vartheta} \qquad ; x \geq 0, \alpha, and\ \vartheta > 0\,, \tag{5}$$

is the same as the GB distribution with scale parameter δ. Clearly, for $\delta = 1$, the GB distribution reduces to the Burr distribution. The GB distribution is commonly used in reliability analysis, medical, social, and logical experiments, and actuarial science to model the time-to-failure of a system or component.

The reliability function (RF) and hazard rate function (HRF) for the GB distribution are shown in Figure 2 and given respectively by

$$R(x) = (1 + \frac{x^{\alpha}}{\delta})^{-\vartheta} \qquad ; x \geq 0, \alpha, \vartheta, and\ \delta > 0\,, \tag{6}$$

and,

$$h(x) = \frac{\alpha\,\vartheta}{\delta}\ x^{\alpha-1}(1 + \frac{x^{\alpha}}{\delta})^{-1} \qquad ; x \geq 0, \alpha, \vartheta, and\ \delta > 0\,. \tag{7}$$

The GB $dist^{n}$ will be denoted as GB$(\alpha, \vartheta, \delta)$.

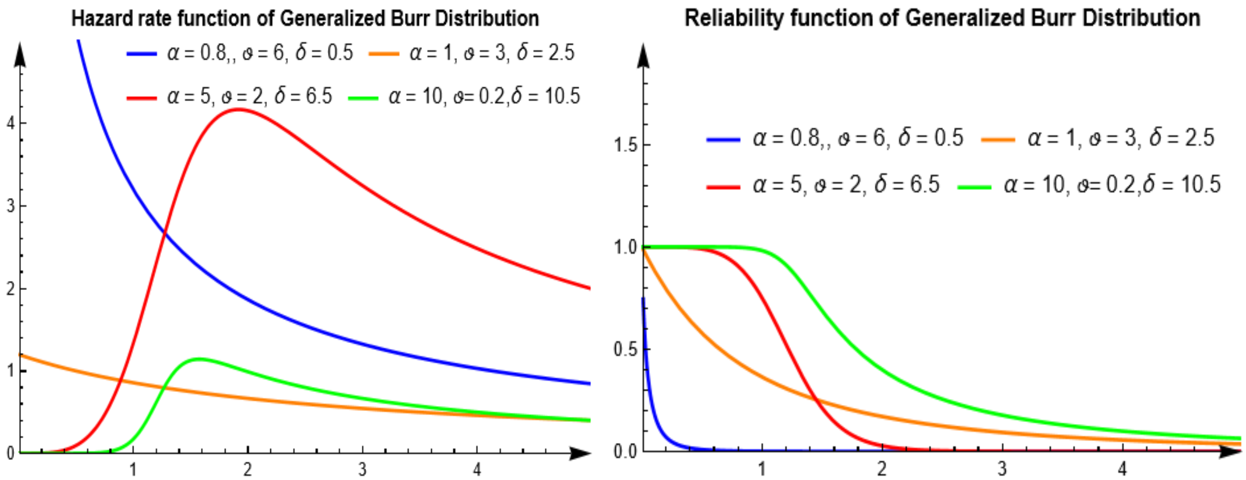


Figure 2. The RF and the HRF of the GB distribution

The GB distribution, like the Weibull distribution, is relevant in a wide range of domains due to its flexible hazard function, which can take on a decrease when $\alpha < 1$, roughly constant $\alpha = 1$, or unimodal shape over time when $\alpha > 1$.

In section 2. We study the bivariate generalized Burr Distribution based on the FGM copula function and obtain several statistical properties of this distribution, such as marginal distributions and conditional CDF, moment generating function, product moment, the joint reliability function, hazard rate function, and reversed hazard rate function of BGB model. In section 3 the estimation of bivariate generalized Burr distribution is considered. In section 4, an analysis of simulated and real data sets is provided. Finally, the paper is concluded in section 5.

2. Bivariate Generalized Burr Distribution Based on FGM

Suppose $\mathrm{X}_1, \mathrm{X}_2$ are two independent random variables such that $\mathrm{X}_1 \sim \mathrm{GB}(\alpha_1, \vartheta_1, \delta_1)$ and $\mathrm{X}_2 \sim \mathrm{GB}(\alpha_2, \vartheta_2, \delta_2)$. According to Sklar's theorem (1959) and Equations (1) and (2), the joint CDF and the joint PDF of bivariate generalized Burr distribution based on FGM copula are as follows:

$$F_{FGMBGB}(x_1, x_2) = [1 - \eta(x_1, \alpha_1, \delta_1, \vartheta_1)][1 - \eta(x_2, \alpha_2, \delta_2, \vartheta_2)][1 + \theta\eta(x_1, \alpha_1, \delta_1, \vartheta_1)\, \eta(x_2, \alpha_2, \delta_2, \vartheta_2)], \quad (8)$$

and

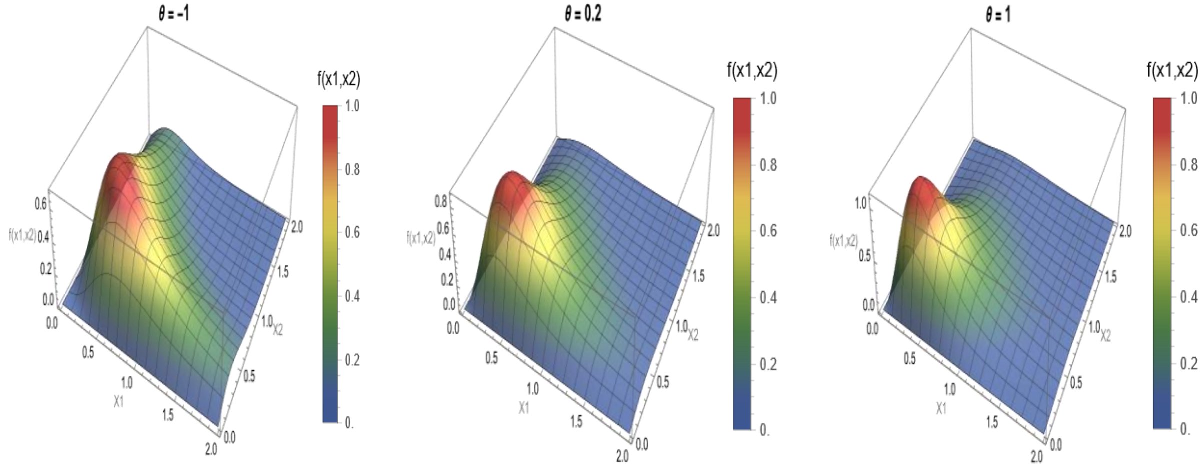
$$f_{FGMBGB}(\mathrm{x}_1, \mathrm{x}_2) = \xi(x_1, \alpha_1, \delta_1, \vartheta_1)\xi(x_2, \alpha_2, \delta_2, \vartheta_2)\{1 + \theta[2\eta(x_1, \alpha_1, \delta_1, \vartheta_1) - 1][2\eta(x_2, \alpha_2, \delta_2, \vartheta_2) - 1]\}, \quad (9)$$

where

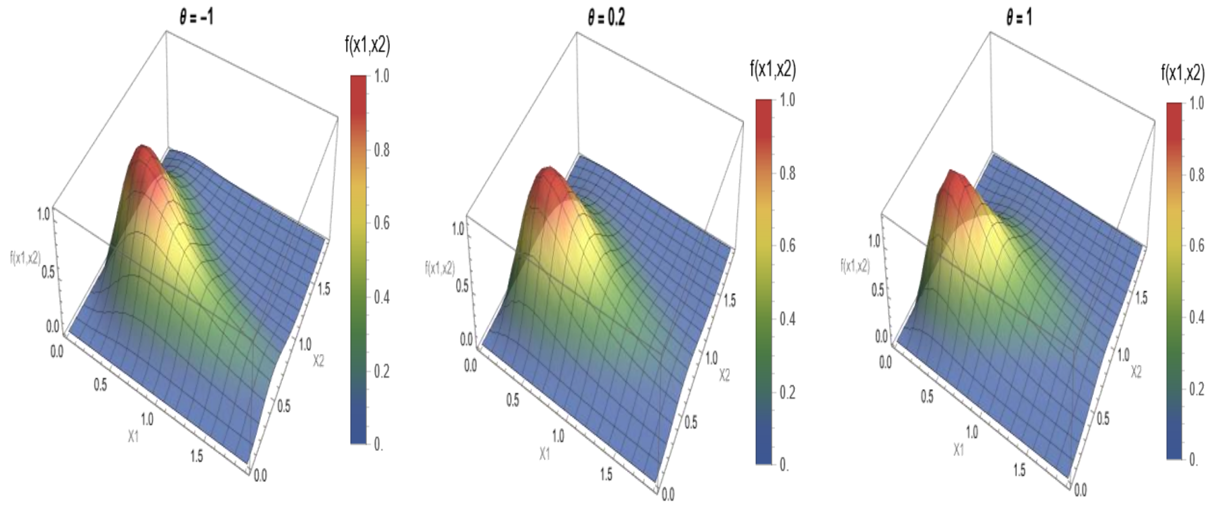
$$\eta(x_i, \alpha_i, \delta_i, \vartheta_i) = \left(1 + \frac{x_i^{\alpha_i}}{\delta_i}\right)^{-\vartheta_i}, \ \xi(x_i, \alpha_i, \delta_i, \vartheta_i) = \frac{\alpha_i\, \vartheta_i}{\delta_i}\, \mathrm{x}_i^{\alpha_i - 1} \left(1 + \frac{\mathrm{x}_i^{\alpha_i}}{\delta_i}\right)^{-\vartheta_i - 1}, i = 1,2.$$

Figure 3 discusses the plots of the joint density function of FGMBGB distribution with parameters values of $\alpha_1, \delta_1, \vartheta_1, \alpha_2, \delta_2, \vartheta_2, and\ \theta$.

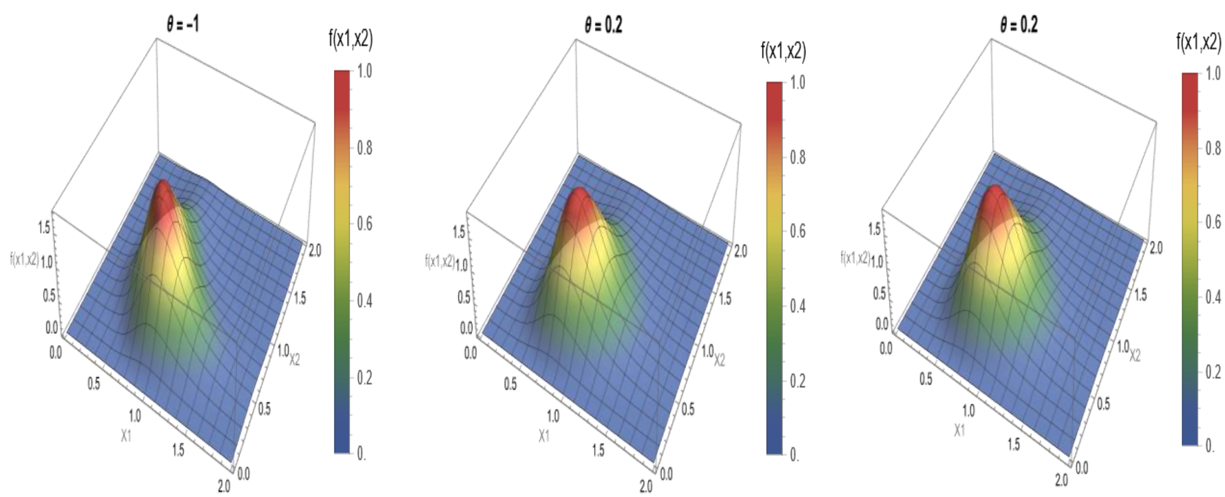
$\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5, \alpha_2 = 2, \delta_2 = 1, \vartheta_2 = 0.5 \; and \; \theta = -1, 0.2 \; and \; 1$



$\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5, \alpha_2 = 3, \delta_2 = 2, \vartheta_2 = 1 \; and \; \theta = -1, 0.2 \; and \; 1$



$\alpha_1 = 5, \delta_1 = 1, \vartheta_1 = 0.5, \alpha_2 = 3, \delta_2 = 2, \vartheta_2 = 1 \; and \; \theta = -1, 0.2 \; and \; 1$



$\alpha_1 = 10, \delta_1 = 1, \vartheta_1 = 0.5, \alpha_2 = 10, \delta_2 = 2, \vartheta_2 = 1 \; and \; \theta = -1, 0.2 \; and \; 1$

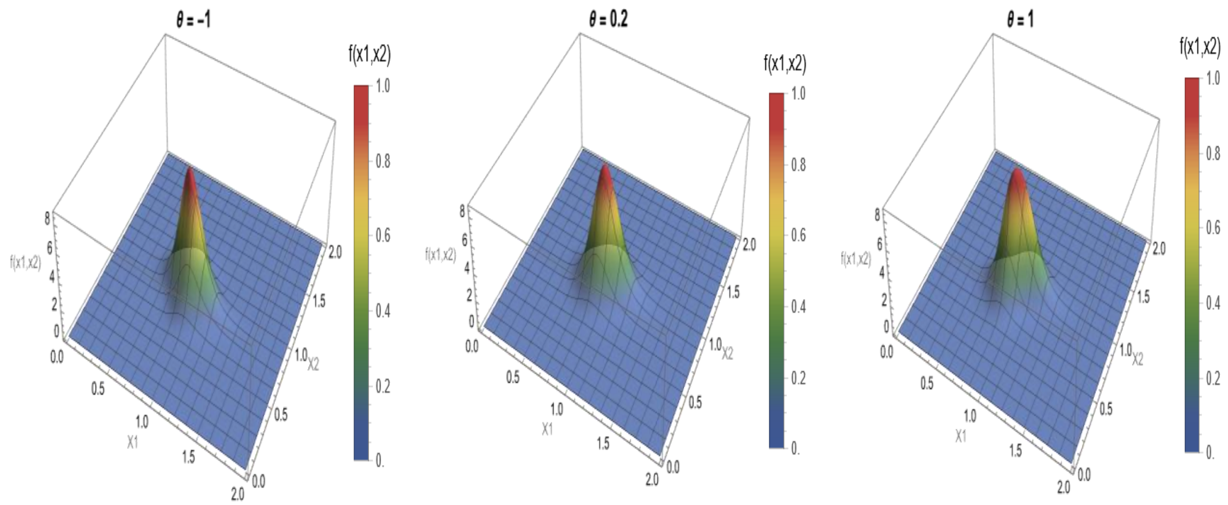


Figure 3. Plots of the joint PDF of FGMBGB distribution

Figure 4 discusses the plots of the joint CDF of FGMBGB distribution with parameters values of $\alpha_1, \delta_1, \vartheta_1, \alpha_2, \delta_2, \vartheta_2, and \; \theta$

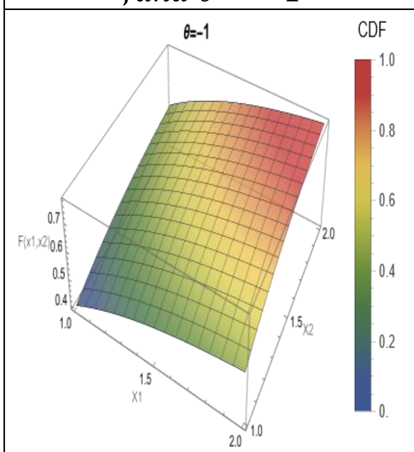
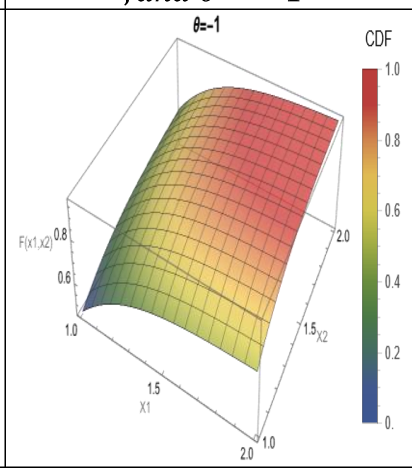
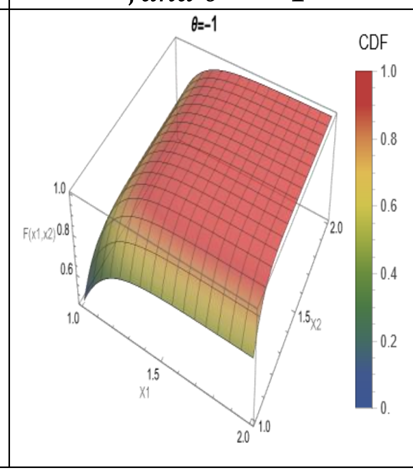
$\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5$ $\alpha_2 = 2, \delta_2 = 1, \vartheta_2 = 0.5$ $, and \; \theta = -1$	$\alpha_1 = 5, \delta_1 = 1, \vartheta_1 = 0.5$ $\alpha_2 = 3, \delta_2 = 2, \vartheta_2 = 1$ $, and \; \theta = -1$	$\alpha_1 = 10, \delta_1 = 1, \vartheta_1 = 0.5$ $\alpha_2 = 10, \delta_2 = 2, \vartheta_2 = 1$ $, and \; \theta = -1$
		

Figure 4. Plots of the joint CDF of FGMBGB distribution

2.1 Marginal Distributions

The marginal density functions for X_1 and X_2 respectively,

$$f(x_1) = \frac{\alpha_1 \vartheta_1}{\delta_1} x_1^{\alpha_1 - 1} (1 + \frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1 - 1} \quad ; \; x_1 > 0, \alpha_1, \vartheta_1, \delta_1 > 0 \quad , \tag{10}$$

$$f(x_2) = \frac{\alpha_2 \vartheta_2}{\delta_2} x_2^{\alpha_2 - 1} (1 + \frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2 - 1} \quad ; x_2 > 0, \alpha_2, \vartheta_2, \delta_2 > 0 \quad . \tag{11}$$

which are generalized Burr distributed, where the marginal distribution X_1 and X_2.

In general

$$f(x_i) = \int_{all x_j} f(x_1,x_2)\,dx_i \quad = f(x_j) \;\; ; i,j = 1,2, i \neq j.$$

2.2 Conditional Distribution

The conditional probability distribution of X_2 given X_1 is given as follows.

$$f(x_2|x_1) = \xi(x_2,\alpha_2,\delta_2,\vartheta_2)\{1+\theta-2\theta[1-\eta(x_1,\alpha_1,\delta_1,\vartheta_1-1)]$$
$$-2\theta[1-\eta(x_2,\alpha_2,\delta_2,\vartheta_2)]+4\theta[1-\eta(x_1,\alpha_1,\delta_1,\vartheta_1)][1-\eta(x_2,\alpha_2,\delta_2,\vartheta_2)], \qquad (12)$$

where

$$\eta(x_i,\alpha_i,\delta_i,\vartheta_i) = (1+\frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i}, \xi(x_i,\alpha_i,\delta_i,\vartheta_i) = \frac{\alpha_i\vartheta_i}{\delta_i}\, x_i^{\alpha_i-1}\,(1+\frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i-1} \;, i = 1,2.$$

The conditional CDF is

$$F(x_2|x_1) = [1-\eta(x_2,\alpha_2,\delta_2,\vartheta_2)]\,\{1+\theta-2\theta[1-\eta(x_1,\alpha_1,\delta_1,\vartheta_1)]$$
$$-2\theta[1-\eta(x_2,\alpha_2,\delta_2,\vartheta_2)]+4\theta[1-\eta(x_1,\alpha_1,\delta_1,\vartheta_1)][1-\eta(x_2,\alpha_2,\delta_2,\vartheta_2)]\}, \qquad (13)$$

where, $\eta(x_i,\alpha_i,\delta_i,\vartheta_i) = (1+\frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i}$, $i = 1,2.$

2.3 Moment Generating Function

If $(X_1, X_2) \sim$ BGB $(\alpha_1, \vartheta_1, \delta_1, \alpha_2, \vartheta_2, \delta_2, \theta)$ with joint pdf given in Equation (9), then, the moment-generating function of BGB distribution is given as follows:
$M_{(x_1,x_2)}(t_1,t_2) =$

$$\sum_{n=0}^{\infty}\frac{\vartheta_1\delta_1^{\frac{n}{\alpha_1}}\,t_1^{\,n}}{n!}B(\frac{n}{\alpha_1}+1,\vartheta_1-\frac{n}{\alpha_1})\sum_{m=0}^{\infty}\frac{\vartheta_2\delta_2^{\frac{m}{\alpha_2}}\,t_2^{\,m}}{m!}B(\frac{m}{\alpha_2}+1,\vartheta_2-\frac{m}{\alpha_2})[1+\theta]$$

$$-2\theta\{\sum_{n=0}^{\infty}\frac{\vartheta_1\delta_1^{\frac{n}{\alpha_1}}\,t_1^{\,n}}{n!}B(\frac{n}{\alpha_1}+1,\vartheta_1-\frac{n}{\alpha_1})\sum_{m=0}^{\infty}\frac{\vartheta_2\delta_2^{\frac{m}{\alpha_2}}\,t_2^{\,m}}{m!}[B(\frac{m}{\alpha_2}+1,\vartheta_2-\frac{m}{\alpha_2})$$

$$-B(\frac{m}{\alpha_2}+1,2\vartheta_2-\frac{m}{\alpha_2})]\}-2\theta\{\sum_{n=0}^{\infty}\frac{\vartheta_1\delta_1^{\frac{n}{\alpha_1}}\,t_1^{\,n}}{n!}[B(\frac{n}{\alpha_1}+1,\vartheta_1-\frac{n}{\alpha_1})-B(\frac{n}{\alpha_1}+1,2\vartheta_1-\frac{n}{\alpha_1})]$$

$$\sum_{m=0}^{\infty}\frac{\vartheta_2\delta_2^{\frac{m}{\alpha_2}}\,t_2^{\,m}}{m!}B(\frac{m}{\alpha_2}+1,\vartheta_2-\frac{m}{\alpha_2})\}+4\theta\{\sum_{n=0}^{\infty}\frac{\vartheta_1\delta_1^{\frac{n}{\alpha_1}}\,t_1^{\,n}}{n!}[B(\frac{n}{\alpha_1}+1,\vartheta_1-\frac{n}{\alpha_1})$$

$$-B(\frac{n}{\alpha_1}+1,2\vartheta_1-\frac{n}{\alpha_1})]\sum_{m=0}^{\infty}\frac{\vartheta_2\delta_2^{\frac{m}{\alpha_2}}\,t_2^{\,m}}{m!}[B(\frac{m}{\alpha_2}+1,\vartheta_2-\frac{m}{\alpha_2})-B(\frac{m}{\alpha_2}+1,2\vartheta_2-\frac{m}{\alpha_2}]\}, \qquad (14)$$

where $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is the beta function, $\vartheta_1 > \frac{n}{\alpha_1}$, and $\vartheta_2 > \frac{m}{\alpha_2}$.

2.4 Product Moments

If the random variable (X_1,X_2) is distributed as FGMBGB, then its $r^{th}$and s^{th}moments around zero can be expressed as follows.

$$E\,(X_1^r X_2^s) = \vartheta_1 \delta_1^{\frac{r}{\alpha_1}}\, B\left(\frac{r}{\alpha_1}+1, \vartheta_1 - \frac{r}{\alpha_1}\right) \vartheta_2 \delta_2^{\frac{s}{\alpha_2}}\, B\left(\frac{s}{\alpha_2}+1, \vartheta_2 - \frac{s}{\alpha_2}\right)[1+\theta]$$

$$-2\theta\{\vartheta_1 \delta_1^{\frac{r}{\alpha_1}} B(\frac{r}{\alpha_1}+1, \vartheta_1 - \frac{r}{\alpha_1}) \vartheta_2 \delta_2^{\frac{s}{\alpha_2}}\, [B(\frac{s}{\alpha_2}+1, \vartheta_2 - \frac{s}{\alpha_2}) - B(\frac{s}{\alpha_2}+1, 2\vartheta_2 - \frac{s}{\alpha_2})]\}$$

$$-2\theta\{\vartheta_1 \delta_1^{\frac{r}{\alpha_1}} \left[B\left(\frac{r}{\alpha_1}+1, \vartheta_1 - \frac{r}{\alpha_1}\right) - B\left(\frac{r}{\alpha_1}+1, 2\vartheta_1 - \frac{r}{\alpha_1}\right)\right] \vartheta_2 \delta_2^{\frac{s}{\alpha_2}}\, B(\frac{s}{\alpha_2}+1, \vartheta_2 - \frac{s}{\alpha_2})\}$$

$$+\,4\theta\{\vartheta_1 \delta_1^{\frac{r}{\alpha_1}} [B(\frac{r}{\alpha_1}+1, \vartheta_1 - \frac{r}{\alpha_1}) - B(\frac{r}{\alpha_1}+1, 2\vartheta_1 - \frac{r}{\alpha_1})][B(\frac{s}{\alpha_2}+1, \vartheta_2 - \frac{s}{\alpha_2})$$

$$-B(\frac{s}{\alpha_2}+1, 2\vartheta_2 - \frac{s}{\alpha_2})]\}, \qquad (15)$$

where $B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$ is the beta function, $\vartheta_1 > \frac{r}{\alpha_1}$, and $\vartheta_2 > \frac{s}{\alpha_2}$.

2.5 Correlation Method

Fredricks and Nelsen (2007) discussed the relationship between Spearman's rho and Kendall's Tau. Since these two metrics assess different facets of the dependent structure,

The Spearman's of FGM *C is* as follows

$$\rho_{sperman} = (12 \iint uv(1+\theta(1-u)(1-v))\, du\, dv - 3 = \frac{\theta}{3}\ , \qquad (16)$$

The Kendall's Tau of FGM *C is* as follows

$$\rho_{kendall} = 1 - 4 \iint \frac{\partial c}{\partial u} C(u,v)\ \frac{\partial c}{\partial x_2} C(u,v) du\, dv = \frac{2}{9}\theta\ . \qquad (17)$$

Figure 5 discusses the correlation coefficient of FGM copula with various values of the copula parameter.

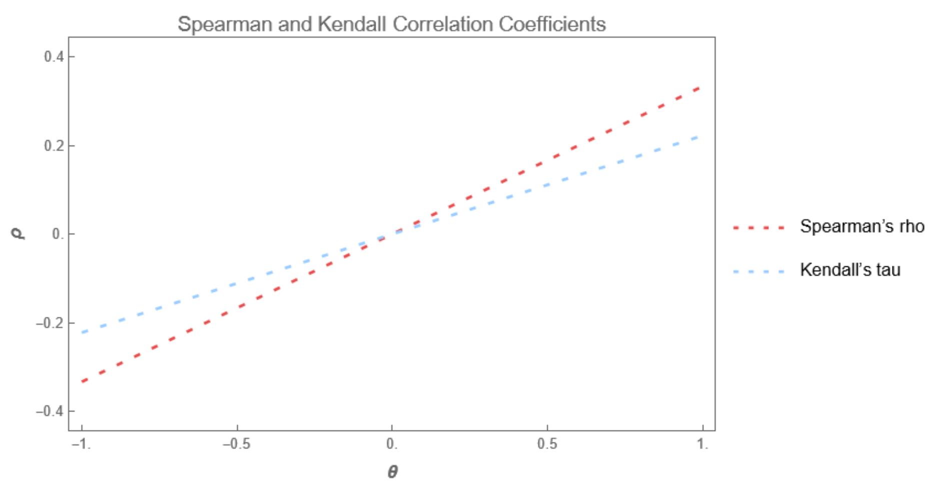


Figure 5. Correlation coefficient of FGM copula

The correlation coefficient ρ measures the strength and direction of a linear relationship between the two variables, where -1 $\leq \theta \leq$ 1.

2.6 Reliability Function

Nelson (2006) introduced the expression of the joint reliability function for copula as follows:

$$\bar{H}(x_1, x_2) = 1 - F(x_1) - F(x_2) + H(x_1, x_2)$$

$$R(x_1, x_2) = \bar{F}(x_1) + \bar{F}(x_2) - 1 + C(F(x_1), F(x_2))$$

$$R_{FGMBGB}(x_1, x_2) = \eta(x_1, \alpha_1, \delta_1, \vartheta_1) + \eta(x_2, \alpha_2, \delta_2, \vartheta_2) - 1 + [1 + \theta(1 - \eta(x_1, \alpha_1, \delta_1, \vartheta_1))(1 - \eta(x_2, \alpha_2, \delta_2, \vartheta_2))] \tag{18}$$

where $\eta(x_i, \alpha_i, \delta_i, \vartheta_i) = (1 + \frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i}$, $i = 1,2.$

Figure 6 discusses the plots of the joint reliability function of FGMBGB distribution with parameters values as $\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5, \alpha_2 = 2, \delta_2 = 1, \vartheta_2 = 0.5\ and\ \theta = -1,0.2$

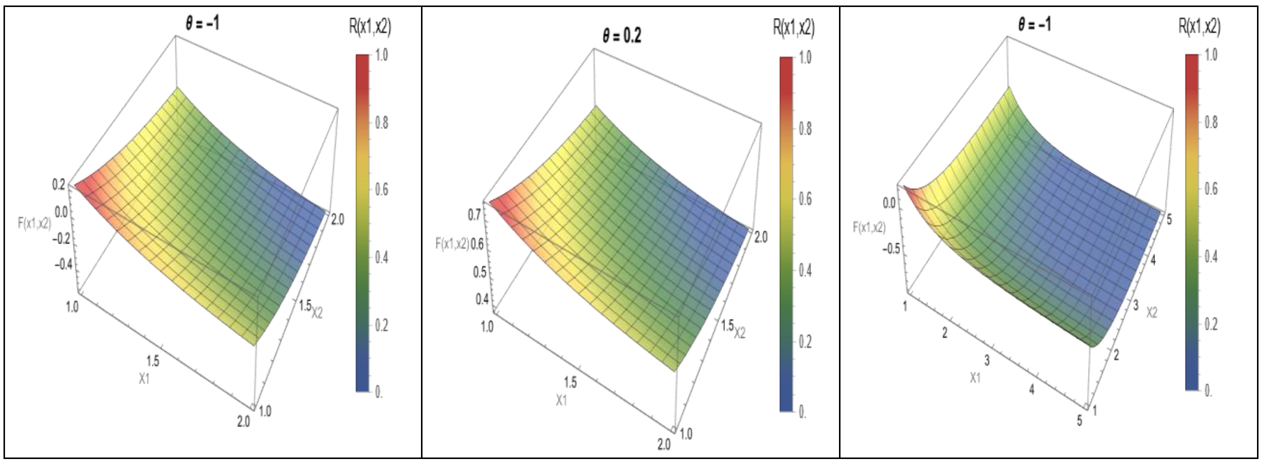


Figure 6. Plots of the joint reliability function of the FGMBGB distribution

2.7 The Hazard Rate Function

Basu (1971) defined the bivariate hazard rate function as follows

$$h(x_1, x_2) = \frac{f(x_1,x_2)}{R(x_1,x_2)}$$

From the definition of $f(x_1, x_2), R(x_1, x_2)$ shown in Equations. (9), (18), we can obtain

$$h(x_1, x_2) = \xi(x_1, \alpha_1, \delta_1, \vartheta_1)\xi(x_2, \alpha_2, \delta_2, \vartheta_2)(1 + \theta(2\eta(x_1, \alpha_1, \delta_1, \vartheta_1) - 1)\ (2\eta(x_2, \alpha_2, \delta_2, \vartheta_2) - 1\ ))/ \eta(x_1, \alpha_1, \delta_1, \vartheta_1) + \eta(x_2, \alpha_2, \delta_2, \vartheta_2) - 1 + \left[1 + \theta\left(1 - \eta(x_1, \alpha_1, \delta_1, \vartheta_1)\right)\left(1 - \eta(x_2, \alpha_2, \delta_2, \vartheta_2)\right)\right], \tag{19}$$

where $\eta(x_i, \alpha_i, \delta_i, \vartheta_i) = (1 + \frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i}$, $\xi(x_i, \alpha_i, \delta_i, \vartheta_i) = \frac{\alpha_i \vartheta_i}{\delta_i}\ x_i^{\alpha_i - 1}\ (1 + \frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i - 1}\ i = 1,2.$

Figure 7 discusses the plots of the joint hazard function of FGMBGB distribution with parameters values as $\alpha_1, \delta_1, \vartheta_1, \alpha_2, \delta_2, \vartheta_2 and\ \theta$

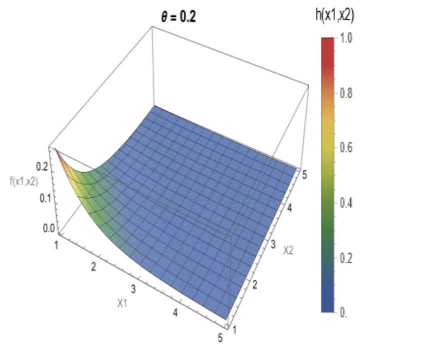
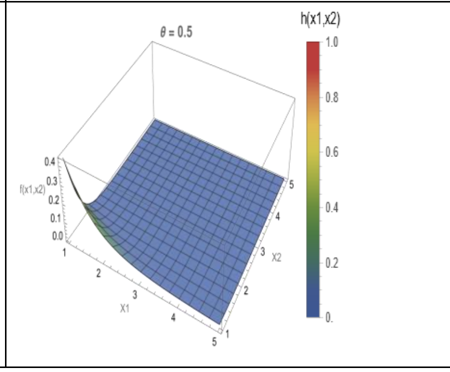
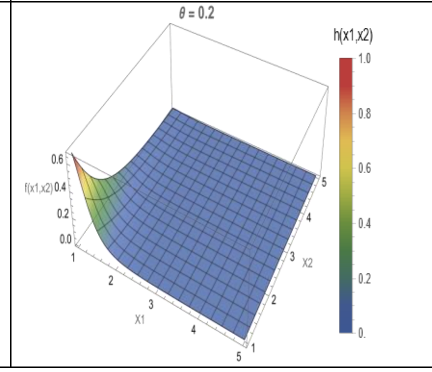
$\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5$ $, \alpha_2 = 2, \delta_2 = 1, \vartheta_2 = 0.5,$ $and\ \theta = 0.2$	$\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5$ $, \alpha_2 = 3, \delta_2 = 1, \vartheta_2 = 2,$ $and\ \theta = 0.5$	$\alpha_1 = 5, \delta_1 = 1, \vartheta_1 = 0.5$ $, \alpha_2 = 3, \delta_2 = 1, \vartheta_2 = 2,$ $and\ \theta = 0.2$
		

Figure 7. Plots of the hazard rate function of FGMBGB distribution

2.8 Reversed Hazard Rate

Domma (2011) defined the bivariate reversed hazard rate (RHR) as the density-to-distribution function ratio.

$$RHR_{FGMBGB}(x) = \frac{f_{FGMBGB}(x_1, x_2)}{F_{FGMBGB}(x_1, x_2)} =$$

$$\xi(x_1, \alpha_1, \delta_1, \vartheta_1)\xi(x_2, \alpha_2, \delta_2, \vartheta_2)(1 + \theta(2\eta(x_1, \alpha_1, \delta_1, \vartheta_1) - 1)\ (2\eta_2(x_2, \alpha_2, \delta_2, \vartheta_2) - 1\ ))/$$
$$(1 - \eta(x_1, \alpha_1, \delta_1, \vartheta_1))(1 - \eta(x_2, \alpha_2, \delta_2, \vartheta_2))(1 + \theta\eta(x_1, \alpha_1, \delta_1, \vartheta_1)\ \eta(x_2, \alpha_2, \delta_2, \vartheta_2))\ , \quad (20)$$

where

$$\eta(x_i, \alpha_i, \delta_i, \vartheta_i) = (1 + \frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i}, \xi(x_i, \alpha_i, \delta_i, \vartheta_i) = \frac{\alpha_i\, \vartheta_i}{\delta_i}\ x_i^{\alpha_i - 1}\ (1 + \frac{x_i^{\alpha_i}}{\delta_i})^{-\vartheta_i - 1}\ , i = 1,2.$$

Figure 8 discusses the plots of the reversed hazard rate function of FGMBGB distribution with parameters values as $\alpha_1, \delta_1, \vartheta_1, \alpha_2, \delta_2, \vartheta_2 and\ \theta$

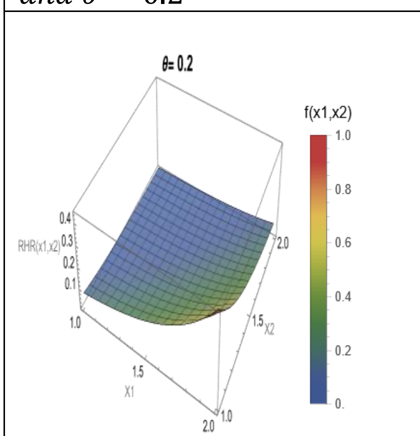
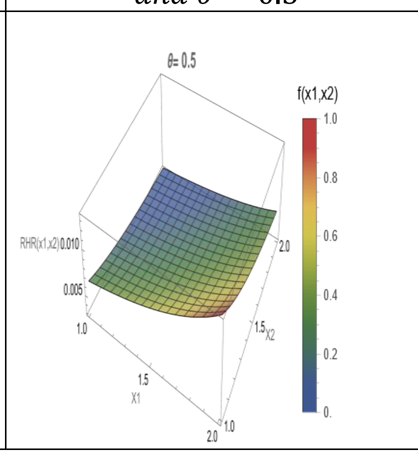
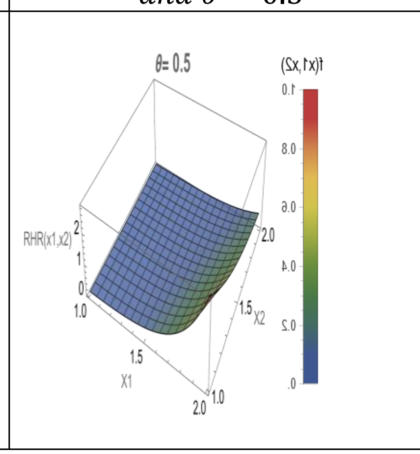
$\alpha_1 = 2, \delta_1 = 1, \vartheta_1 = 0.5$ $, \alpha_2 = 2, \delta_2 = 1, \vartheta_2 = 0.5,$ $and\ \theta = 0.2$	$\alpha_1 = 0.2, \delta_1 = 0.1, \vartheta_1 = 0.5$ $, \alpha_2 = 0.3, \delta_2 = 0.1, \vartheta_2 = 0.2$ $and\ \theta = 0.5$	$\alpha_1 = 5, \delta_1 = 1, \vartheta_1 = 0.5$ $, \alpha_2 = 3, \delta_2 = 1, \vartheta_2 = 2,$ $and\ \theta = 0.5$
		

Figure 8. Plots of the reversed hazard rate function of FGMBGB distribution

3. Maximum Likelihood Estimation (MLE) Based on Copula

In this section, we introduced the estimation of the unknown parameter for BGB using maximum likelihood estimation. Suppose that $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ be a random sample from BGB with unknown parameters $\alpha_1, \vartheta_1, \delta_1, \alpha_2, \vartheta_2, \delta_2, and\ \theta$. The likelihood function of BGB distribution based on different copulas are as follows:

$$L(\Theta) = \Pi_{i=1}^{n} f(x_{1i}) g(x_{2i}) c\big(F(x_{1i}), G\,(x_{2i})\big) \quad , \qquad i = 1, \ldots, n \tag{21}$$

The natural logarithm of the likelihood function for the FGMBGB distribution is defined as

$$l(\Theta) = nlog\alpha_1 + n\,log\vartheta_1 - n\,log\delta_1 + (\alpha_1 - 1)\Sigma_{i=1}^{n} log x_{1i} - (\vartheta_1 + 1)\,\Sigma_{i=1}^{n} A(x_{1i}, \alpha_1, \delta_1)$$
$$+ nlog\alpha_2 + n\,log\vartheta_2 - n\,log\delta_2 + (\alpha_2 - 1)\Sigma_{i=1}^{n} log x_{2i} - (\vartheta_2 + 1)\,\Sigma_{i=1}^{n} A(x_{2i}, \alpha_2, \delta_2)$$
$$+\Sigma_{i=1}^{n}\, log\{1 + \theta[2\eta(x_{1i}, \alpha_1, \delta_1, \vartheta_1) - 1][2\eta(x_{2i}, \alpha_2, \delta_2, \vartheta_2) - 1]\}. \tag{22}$$

where

$$\Theta = (\alpha_1, \vartheta_1, \delta_1, \alpha_2, \vartheta_2, \delta_2, \theta),\ A\,(x_j, \alpha_j, \delta_j) = log(1 + \frac{{x_j}^{\alpha_j}}{\delta_j}),$$

and

$$\eta\big(x_j, \alpha_j, \delta_j, \vartheta_j\big) = (1 + \frac{x_j^{\alpha_j}}{\delta_j})^{-\vartheta_j} \ , j = 1,2.$$

By obtaining the first derivatives and equating them to zero. We can get the likelihood equations and hence solve them numerically to estimate the parameters.

The First Derivatives of $l(\Theta)$ with respect to each parameter individually, are given as

$$\frac{\partial l(\Theta)}{\partial\,\alpha_1} = \frac{n}{\alpha_1} + \Sigma_{i=1}^{n} log x_{1i} - (\vartheta_1 + 1)\Sigma_{i=1}^{n} B(x_{1i}; \alpha_1, \vartheta_1, \delta_1)$$
$$+ \Sigma_{i=1}^{n} D(x_{1i}, x_{2i}; \Theta), \tag{23}$$

$$\frac{\partial l(\Theta)}{\partial\,\alpha_2} = \frac{n}{\alpha_2} + \Sigma_{i=1}^{n} log x_{2i} - (\vartheta_2 + 1)\Sigma_{i=1}^{n} \mathrm{Z}\,(x_{2i}; \alpha_2, \vartheta_2, \delta_2)$$
$$+ \Sigma_{i=1}^{n} D(x_{2i}, x_{1i}; \Theta), \tag{24}$$

$$\frac{\partial l(\Theta)}{\partial\,\vartheta_1} = \frac{n}{\vartheta_1} - \Sigma_{i=1}^{n} A(x_{1i}; \alpha_1, \delta_1)$$
$$+ \Sigma_{i=1}^{n} \mathrm{H}(x_{1i}, x_{2i}; \Theta) \tag{25}$$

$$\frac{\partial l(\Theta)}{\partial\,\vartheta_2} = \frac{n}{\vartheta_2} - \Sigma_{i=1}^{n} A(x_{2i}, \alpha_2, \delta_2)$$
$$+ \Sigma_{i=1}^{n} \mathrm{H}(x_{2i}, x_{1i}; \Theta) \tag{26}$$

$$\frac{\partial l(\Theta)}{\partial\,\delta_1} = \frac{-n}{\delta_1} - (\vartheta_1 + 1)\Sigma_{i=1}^{n} c(x_{1i}; \alpha_1, \vartheta_1, \delta_1) + \Sigma_{i=1}^{n} d(x_{1i}, x_{2i}, \Theta), \tag{27}$$

$$\frac{\partial l(\Theta)}{\partial\,\delta_2} = \frac{-n}{\delta_2} - (\vartheta_2 + 1)\Sigma_{i=1}^{n} \mathrm{M}(x_{2i}; \alpha_2, \vartheta_2, \delta_2) + \Sigma_{i=1}^{n} d(x_{2i}, x_{1i}, \Theta), \tag{28}$$

and

$$\frac{\partial l(\Theta)}{\partial\,\theta} = \Sigma_{i=1}^{n} E(x_{1i}, x_{2i}, \Theta), \tag{29}$$

where

$\Theta = (\alpha_1, \vartheta_1, \delta_1, \alpha_2, \vartheta_2, \delta_2, \theta)$, $B(x_1; \alpha_1, \delta_1) = \frac{{x_1}^{\alpha_1} log x_1}{\delta_1 + {x_1}^{\alpha_1}}$, $\mathrm{Z}(x_2; \alpha_2, \delta_2) = \frac{{x_2}^{\alpha_2} log x_2}{\delta_2 + {x_2}^{\alpha_2}}$

$c(x_1; \alpha_1, \delta_1) = \frac{-{x_1}^{\alpha_1}}{\delta_1(\delta_1 + {x_1}^{\alpha_1})}$, $\mathrm{M}(x_2; \alpha_2, \delta_2) = \frac{-{x_2}^{\alpha_2}}{\delta_2(\delta_2 + {x_2}^{\alpha_2})}$

$$D(x_1, x_2; \Theta) = \frac{\theta[2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1][-2\vartheta_1(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\frac{x_1^{\alpha_1}}{\delta_1} log x_1]}{\{1+\theta[2\eta(x_1,\alpha_1,\delta_1,\vartheta_1)-1][2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1]\}},$$

$$d(x_1, x_2; \Theta) = \frac{\theta[2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1][2\vartheta_1(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\frac{x_1^{\alpha_1}}{{\delta_1}^2}]}{\{1+\theta[2\eta(x_1,\alpha_1,\delta_1,\vartheta_1)-1][2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1]\}},$$

$$\mathrm{H}(x_1, x_2; \Theta) = \frac{[2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1][-2\theta log(1+\frac{x_1^{\alpha_1}}{\delta_1})(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}]}{\{1+\theta[2\eta(x_1,\alpha_1,\delta_1,\vartheta_1)-1][2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1]\}},$$

And

$$E(x_1, x_2; \Theta) = \frac{[2\eta(x_1,\alpha_1,\delta_1,\vartheta_1)-1][2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1]}{\{1+\theta[2\eta(x_1,\alpha_1,\delta_1,\vartheta_1)-1][2\eta(x_2,\alpha_2,\delta_2,\vartheta_2)-1]\}}.$$

Asymptotic Confidence Intervals

We can obtain the asymptotic variance-covariance matrix by inverting the information matrix $I(\Theta)$ with elements that are the negatives of the expected values of the second-order derivatives of logarithms of the likelihood function. The variance-covariance matrix is given by Cohen (1965) as follows.

$$V.cov(\widehat{\Theta}) = I^{-1}(\widehat{\Theta}) = \left.\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} & I_{16} & I_{17} \\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25} & I_{26} & I_{27} \\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35} & I_{36} & I_{37} \\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45} & I_{46} & I_{47} \\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55} & I_{56} & I_{57} \\ I_{61} & I_{62} & I_{63} & I_{64} & I_{65} & I_{66} & I_{67} \\ I_{71} & I_{72} & I_{73} & I_{74} & I_{75} & I_{76} & I_{77} \end{bmatrix}^{-1} \right|_{\Theta=\widehat{\Theta}} \tag{30}$$

where $\Theta = \alpha_1, \alpha_2, \vartheta_1, \vartheta_2, \delta_1, \delta_2, and\ \theta$, $\widehat{\Theta} = \widehat{\alpha_1}, \widehat{\alpha_2}, , \widehat{\vartheta_1}, \widehat{\vartheta_2}, \widehat{\delta_1}, \widehat{\delta_2}, and\ \hat{\theta}$, $and\ \ j = 1{,}2.$

Using Equation (30), under the assumption that some regularity conditions are satisfied. We can get the asymptotic confidence intervals of the parameters of FGMBGB distribution as follows.

$\widehat{\alpha_1} \pm Z_{0.025}\sqrt{I_{11}}$, $\widehat{\alpha_2} \pm Z_{0.025}\sqrt{I_{22}}$, $\widehat{\vartheta_1} \pm Z_{0.025}\sqrt{I_{33}}$, $\widehat{\vartheta_2} \pm Z_{0.025}\sqrt{I_{44}}$,

$\widehat{\delta_1} \pm Z_{0.025}\sqrt{I_{55}}$, $\widehat{\delta_2} \pm Z_{0.025}\sqrt{I_{66}}$ and $\hat{\theta} \pm Z_{0.025}\sqrt{I_{77}}$.

Where $Z_{0.025}$ is the critical value from the standard normal distribution corresponding to the right tail probability $\frac{\gamma}{2}$. I_{ii} is the asymptotic variance of $\widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\vartheta_1}, \widehat{\vartheta_2}, \widehat{\delta_1}, \widehat{\delta_2}\ and\ \hat{\theta}$.

The elements of the observed information matrix are in **Appendix A.**

4. Numerical Study

This section presents a comprehensive analysis of both simulated and real data sets. The analysis aims to demonstrate the effectiveness and applicability of the FGM bivariate generalized Burr distribution by evaluating its performance using economic data and simulated data.

4.1 Simulation of FGMBGB Model

A Monte Carlo simulation is done to study the performance of the FGMBGB model. Nelsen (2006) discussed generating a sample from a specified joint distribution by a conditional distribution method. The joint distribution function is as follows.

$f(x_1, x_2) = f(x_1) f(x_2|x_1)$

By using the following steps, we can generate a bivariate sample by using the conditional approach:

1. Generate u and v independently from a uniform (0,1)
2. Calculate $X_1 = {\delta_1}^{1/\alpha_1} [(1-U)^{-1/\vartheta_1} - 1]^{1/\alpha_1}$, $U = F(X_1)$
3. Set $F(x_2|x_1) = v\ \ .to\ find\ x_2\ by\ numerical\ simulation$
4. Repeat steps 1-3(n). times to obtain (x_{1i}, x_{2i}), $i\ = 1,2, \dots, n$

A Simulation algorithm: the simulation experiments were carried out based on the following data generated from generalized Burr distribution, where X_1, X_2 are distributed as Generalized Burr with shape parameters α_i, ϑ_i, and δ_i, scale parameter, i = 1,2, the values of the parameters $\alpha_1, \alpha_2, \vartheta_1, \vartheta_2, \delta_1, \delta_2 and\ \theta$ is chosen as the following cases for the random variables generating:

Case 1: (α_1 =1.5, ϑ_1 =1, δ_1 =0.78, α_2 =1.5, ϑ_2 =0.4, δ_2 =0.8, θ = 0.5),

Case 2: (α_1 =1.5, ϑ_1 =1, δ_1 =0.78, α_2 =1.5, ϑ_2 =0.4, δ_2 =0.8, θ = 0.3),

Case 3:(α_1 =1.5, ϑ_1 =1, δ_1 =0.78, α_2 =1.5, ϑ_2 =0.4, δ_2 =0.8, θ = - 0.5).

For different sample sizes n = 100, 150, 200, 500, and 1000, with 1000 replications for each case. All computations are obtained based on the R language. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculating the Bias, SE, MSE, the length of the confidence interval (L.CI), and the dependence measure for each case as follows

$$Bias = (\widehat{\Theta} - \Theta),$$

where $\widehat{\Theta}$ is the estimated value of Θ

$$\text{MSE}=Mean\ (\widehat{\Theta} - \Theta)^2$$

and

$$\text{L.}CI = Upper.\ CI - Lower.CI.$$

Additionally, the dependence between X_1 and $X_2$was measured using Kendall's Tau and Spearman Rho coefficients, are calculated based on Equations (16) and (17) for three cases. The results were summarized in Tables 1 to 6 for the dependence measures. The following are some inferences that can be made:

1. The estimates of Bias, SE, MSE, and L.CI decrease with increasing sample size and fixed vector value of Θ. All of them are almost equal in big sample sizes, with less of a difference.

2. For large sample sizes $n \geq 500$, the estimates become stable, with the differences between estimated and true values diminishing, indicating improved statistical performance with larger samples.

3. Kendall's Tau and Spearman's Rho coefficients for the FGMBGB distribution of the Bias, SE, MSE, and L.CI decrease as n increases for actual parameters, reflecting the stability of the estimates and their convergence to the true dependence values. This decrease may also indicate reduced variability in the estimates with larger sample sizes.

4. Changing the value of θ $(0.5, 0.3, -0.5)$ has a significant impact on the accuracy of estimates in small $samples$ $(n < 200)$. For instance, negative values of $\theta = -0.5$ in case 3 increase the variance of the distribution increases, leading to higher Bias and MSE compared to positive values. This highlights the importance of carefully selecting θ, especially in models relying on small samples.

5. The small samples (less than 100) show less accurate results, with an increase in bias, MSE, and L.CI compared to larger samples. Therefore, it would have been better to use samples with a size greater than or equal to 100.

Table 1. MLE of the parameters for $FGMBGB$ distribution Case 1

n		Estimate	MSE	SE	Bias	Lower	Upper	L.CI
100	$\hat{\alpha}_1$	1.543671	0.448259	0.267406	0.59042	1.019556	2.067786	1.04823
	$\hat{\vartheta}_1$	1.063958	0.50708	0.470601	0.348879	0.141579	1.986336	1.844756
	$\hat{\delta}_1$	0.883436	0.917403	0.620726	0.46092	-0.33319	2.100058	2.433244
	$\hat{\alpha}_2$	1.542171	0.53464	0.324549	0.629544	0.906055	2.178286	1.272231
	$\hat{\vartheta}_2$	0.429037	0.403509	0.14774	0.587407	0.139467	0.718607	0.57914
	$\hat{\delta}_2$	0.935373	0.976113	0.529643	0.500853	-0.10273	1.973474	2.076202

n		Estimate	MSE	SE	Bias	Lower	Upper	L.CI
	$\hat{\theta}$	0.470478	0.095761	0.208091	0.220868	0.06262	0.878335	0.815715
	$\hat{\alpha}_1$	1.530808	0.370077	0.207643	0.564388	1.123827	1.937789	0.813962
	$\hat{\vartheta}_1$	1.052961	0.348852	0.366556	0.285783	0.334511	1.771412	1.436901
	$\hat{\delta}_1$	0.84785	0.64999	0.475474	0.382396	-0.08408	1.779779	1.863858
150	$\hat{\alpha}_2$	1.520605	0.396526	0.246718	0.574707	1.037038	2.004171	0.967133
	$\hat{\vartheta}_2$	0.422568	0.358092	0.110116	0.586677	0.206742	0.638395	0.431654
	$\hat{\delta}_2$	0.908696	0.330003	0.397757	0.382118	0.129093	1.688299	1.559206
	$\hat{\theta}$	0.487241	0.060903	0.167206	0.177699	0.159517	0.814965	0.655447
	$\hat{\alpha}_1$	1.510443	0.320008	0.176164	0.53338	1.165162	1.855724	0.690562
	$\hat{\vartheta}_1$	1.049502	0.176017	0.307319	0.232206	0.447157	1.651846	1.20469
	$\hat{\delta}_1$	0.846008	0.321948	0.400313	0.329141	0.061395	1.630621	1.569226
200	$\hat{\alpha}_2$	1.518483	0.363223	0.212684	0.560875	1.101621	1.935344	0.833723
	$\hat{\vartheta}_2$	0.415333	0.359174	0.09234	0.591582	0.234346	0.59632	0.361974
	$\hat{\delta}_2$	0.874223	0.237574	0.330908	0.34243	0.225643	1.522803	1.29716
	$\hat{\theta}$	0.488051	0.044752	0.144732	0.151898	0.204377	0.771726	0.567349
	$\hat{\alpha}_1$	1.49756	0.267354	0.104787	0.506136	1.292175	1.702942	0.410766
	$\hat{\vartheta}_1$	1.03325	0.055225	0.178017	0.146833	0.684339	1.382167	0.697828
	$\hat{\delta}_1$	0.82647	0.122127	0.231966	0.253535	0.37182	1.281129	0.909308
500	$\hat{\alpha}_2$	1.49209	0.275614	0.128441	0.50869	1.240345	1.743835	0.50349
	$\hat{\vartheta}_2$	0.41269	0.35133	0.05683	0.589915	0.301304	0.524077	0.222774
	$\hat{\delta}_2$	0.853023	0.102274	0.203015	0.243653	0.455113	1.250932	0.795819
	$\hat{\theta}$	0.499033	0.017755	0.091243	0.096411	0.320197	0.677868	0.357671
	$\hat{\alpha}_1$	1.495318	0.255523	0.07352	0.500068	1.351218	1.639418	0.2882
	$\hat{\vartheta}_1$	1.02474	0.027608	0.122899	0.109622	0.783857	1.265623	0.481766
	$\hat{\delta}_1$	0.813838	0.079812	0.159735	0.230867	0.500757	1.126918	0.626161
1000	$\hat{\alpha}_2$	1.485741	0.252207	0.090159	0.493951	1.309029	1.662453	0.353424
	$\hat{\vartheta}_2$	0.410643	0.35051	0.039828	0.590676	0.332581	0.488706	0.156125
	$\hat{\delta}_2$	0.841124	0.064759	0.14183	0.21032	0.563137	1.119112	0.555975
	$\hat{\theta}$	0.496965	0.008764	0.064655	0.067474	0.370242	0.623688	0.253446

Table 2. MLE of the parameters for $FGMBGB$ distribution in Case 2

n		Estimate	MSE	SE	Bias	Lower	Upper	L.CI
	$\hat{\alpha}_1$	1.541671	0.444417	0.266224	0.593814	1.019872	2.063469	1.043597
	$\hat{\vartheta}_1$	1.091374	0.712627	0.501906	0.375383	0.107639	2.075109	1.96747
	$\hat{\delta}_1$	0.904514	1.379734	0.657866	0.486404	-0.3849	2.193932	2.578836
100	$\hat{\alpha}_2$	1.550255	0.528584	0.323671	0.634276	0.91586	2.184649	1.26879
	$\hat{\vartheta}_2$	0.428249	0.37375	0.145289	0.587621	0.143483	0.713014	0.569531
	$\hat{\delta}_2$	0.932875	0.604553	0.525901	0.473524	-0.09789	1.963641	2.061532
	$\hat{\theta}$	0.286645	0.18027	0.24439	0.342336	-0.19236	0.765649	0.958008
	$\hat{\alpha}_1$	1.508657	0.343758	0.207815	0.541616	1.10134	1.915975	0.814635
	$\hat{\vartheta}_1$	1.092676	0.318591	0.3934	0.308485	0.321612	1.86374	1.542129
	$\hat{\delta}_1$	0.902356	0.55411	0.517021	0.384115	-0.11101	1.915718	2.026723
150	$\hat{\alpha}_2$	1.526282	0.412526	0.252659	0.584447	1.03107	2.021494	0.990424
	$\hat{\vartheta}_2$	0.426072	0.355873	0.112931	0.584251	0.204728	0.647416	0.442688
	$\hat{\delta}_2$	0.924498	0.370751	0.410532	0.412856	0.119856	1.729139	1.609283
	$\hat{\theta}$	0.289548	0.129055	0.20155	0.2951	-0.10549	0.684586	0.790076
	$\hat{\alpha}_1$	1.514526	0.327985	0.180808	0.538139	1.160142	1.86891	0.708767
	$\hat{\vartheta}_1$	1.066015	0.230487	0.327618	0.263384	0.423883	1.708147	1.284263
200	$\hat{\delta}_1$	0.876657	0.412399	0.431797	0.348411	0.030336	1.722978	1.692642
	$\hat{\alpha}_2$	1.518868	0.36159	0.212996	0.559551	1.101395	1.936341	0.834946
	$\hat{\vartheta}_2$	0.416316	0.358845	0.093522	0.590924	0.233013	0.599618	0.366605

n		Estimate	MSE	SE	Bias	Lower	Upper	L.CI
	$\hat{\delta}_2$	0.88626	0.250895	0.339044	0.348886	0.221733	1.550787	1.329053
	$\hat{\theta}$	0.28184	0.112757	0.176059	0.284501	-0.06324	0.626916	0.690152
500	$\hat{\alpha}_1$	1.48822	0.25927	0.10664	0.49753	1.279198	1.6972559	0.41805
	$\hat{\vartheta}_1$	1.04710	0.06193	0.18690	0.15450	0.680775	1.4134324	0.73265
	$\hat{\delta}_1$	0.84368	0.12509	0.24446	0.24308	0.364536	1.3228382	0.95830
	$\hat{\alpha}_2$	1.49545	0.27951	0.12993	0.51208	1.240778	1.7501366	0.50935
	$\hat{\vartheta}_2$	0.41133	0.35295	0.05707	0.59127	0.299476	0.5231971	0.22372
	$\hat{\delta}_2$	0.85318	0.10453	0.20510	0.24651	0.451185	1.2551835	0.80399
	$\hat{\theta}$	0.30128	0.06501	0.11000	0.22972	0.085680	0.5168834	0.43120
1000	$\hat{\alpha}_1$	1.48380	0.244675	0.073814	0.489055	1.33913	1.628483	0.289353
	$\hat{\vartheta}_1$	1.048136	0.032486	0.129475	0.122302	0.794364	1.301907	0.507542
	$\hat{\delta}_1$	0.84372	0.074995	0.16914	0.211804	0.512207	1.175234	0.663027
	$\hat{\alpha}_2$	1.491484	0.258672	0.090797	0.500335	1.313521	1.669447	0.355926
	$\hat{\vartheta}_2$	0.410378	0.350962	0.039944	0.591051	0.332088	0.488669	0.156581
	$\hat{\delta}_2$	0.843869	0.067929	0.143098	0.216731	0.563397	1.12434	0.560943
	$\hat{\theta}$	0.307719	0.04964	0.077513	0.2088	0.155793	0.459645	0.303852

Table 3. MLE of the parameters for $FGMBGB$ distribution in Case 3

n		Estimate	MSE	SE	Bias	Lower	Upper	L.CI
100	$\hat{\alpha}_1$	1.485164	0.46491	0.296947	0.538392	0.903149	2.06718	1.164031
	$\hat{\vartheta}_1$	1.218228	1.956259	0.743328	0.475516	-0.2387	2.675151	2.913847
	$\hat{\delta}_1$	1.079754	3.378249	0.995535	0.552145	-0.87149	3.031003	3.902496
	$\hat{\alpha}_2$	1.547629	0.613219	0.342041	0.650579	0.877228	2.218029	1.340801
	$\hat{\vartheta}_2$	0.439833	0.374354	0.15871	0.580221	0.128762	0.750905	0.622143
	$\hat{\delta}_2$	0.990244	0.826541	0.582136	0.556378	-0.15074	2.13123	2.281972
	$\hat{\theta}$	-0.52236	0.993108	0.358228	0.9804962	-0.91449	0.079762	0.994252
150	$\hat{\alpha}_1$	1.465748	0.331657	0.233617	0.502352	1.00786	1.923637	0.915777
	$\hat{\vartheta}_1$	1.21955	1.137018	0.610376	0.425871	0.023214	2.415887	2.392673
	$\hat{\delta}_1$	1.076316	2.007849	0.821849	0.479002	-0.53451	2.68714	3.221649
	$\hat{\alpha}_2$	1.523107	0.447702	0.268889	0.585887	0.996084	2.05013	1.054046
	$\hat{\vartheta}_2$	0.432832	0.360176	0.126482	0.579853	0.184928	0.680736	0.495808
	$\hat{\delta}_2$	0.951875	0.531141	0.461746	0.456495	0.046853	1.856897	1.810044
	$\hat{\theta}$	-0.47712	0.951057	0.282786	0.969485	-0.90138	0.077143	0.978523
200	$\hat{\alpha}_1$	1.463649	0.308903	0.199896	0.493744	1.071853	1.855446	0.783594
	$\hat{\vartheta}_1$	1.18972	0.920512	0.494571	0.376683	0.220362	2.159078	1.938717
	$\hat{\delta}_1$	1.02887	1.649588	0.663382	0.418482	-0.27136	2.329099	2.600459
	$\hat{\alpha}_2$	1.511854	0.387241	0.224118	0.568922	1.072583	1.951124	0.878541
	$\hat{\vartheta}_2$	0.429931	0.351968	0.104793	0.580754	0.224538	0.635325	0.410787
	$\hat{\delta}_2$	0.92681	0.364843	0.380228	0.405332	0.181563	1.672057	1.490495
	$\hat{\theta}$	-0.44852	0.931181	0.240897	0.986012	-0.92068	0.023641	0.944316
500	$\hat{\alpha}_1$	1.448268	0.26295	0.132649	0.466582	1.188276	1.70826	0.519984
	$\hat{\vartheta}_1$	1.183742	0.479069	0.332063	0.328967	0.532899	1.834586	1.301687
	$\hat{\delta}_1$	1.022492	0.861333	0.449851	0.358144	0.140785	1.9042	1.763415
	$\hat{\alpha}_2$	1.48682	0.284183	0.135169	0.514093	1.221889	1.751751	0.529862
	$\hat{\vartheta}_2$	0.425419	0.341502	0.064151	0.579754	0.299683	0.551156	0.251473
	$\hat{\delta}_2$	0.91181	0.165618	0.234578	0.30518	0.452036	1.371583	0.919546
	$\hat{\theta}$	-0.43192	0.92183	0.153119	0.947767	-0.73203	-0.13181	0.600226
1000	$\hat{\alpha}_1$	1.436387	0.211304	0.087807	0.448507	1.264286	1.608489	0.344203
	$\hat{\vartheta}_1$	1.191761	0.19958	0.220849	0.301948	0.758897	1.624625	0.865728
	$\hat{\delta}_1$	1.04048	0.314396	0.301639	0.31307	0.449268	1.631693	1.182425

	$\hat{\alpha}_2$	1.492875	0.268887	0.094356	0.509659	1.307937	1.677812	0.369875
	$\hat{\vartheta}_2$	0.418847	0.343443	0.043296	0.584314	0.333986	0.503708	0.169722
	$\hat{\delta}_2$	0.883476	0.095108	0.157813	0.259634	0.574163	1.192788	0.618626
	$\hat{\theta}$	-0.43439	0.902565	0.108343	0.943818	-0.64674	-0.22204	0.424703

Table 4. Dependence measures of the parameters for $FGMBGB$ distribution in Case 1

n		MSE	SE	Bias	L.CI
100	ρ_s	0.031920	0.069363	0.073622	0.271905
	ρ_τ	0.021280	0.046242	0.049081	0.18127
150	ρ_s	0.020301	0.055735	0.059233	0.218482
	ρ_τ	0.013534	0.037156	0.039488	0.145654
200	ρ_s	0.014917	0.048244	0.050632	0.189116
	ρ_τ	0.009944	0.032162	0.033755	0.126077
500	ρ_s	0.0059183	0.030414	0.032137	0.119223
	ρ_τ	0.0039455	0.020276	0.021424	0.079482
1000	ρ_s	0.0029213	0.021551	0.022491	0.084482
	ρ_τ	0.0019475	0.014367	0.014994	0.056321

Table 5. Dependence measures of the parameters for $FGMBGB$ distribution in Case 2

n		MSE	SE	Bias	L.CI
100	ρ_s	0.06009	0.081463	0.114112	0.319336
	ρ_τ	0.04006	0.054308	0.076074	0.212890
150	ρ_s	0.043018	0.06718	0.09836	0.263358
	ρ_τ	0.028678	0.044788	0.06557	0.175572
200	ρ_s	0.037585	0.058686	0.094833	0.23005
	ρ_τ	0.025057	0.039124	0.063222	0.15336
500	ρ_s	0.02167	0.03666	0.07657	0.143733
	ρ_τ	0.01444	0.02444	0.051048	0.095822
1000	ρ_s	0.01654	0.02583	0.0696	0.10128
	ρ_τ	0.01103	0.01722	0.0464	0.067522

Table 6. Dependence measures of the parameters for $FGMBGB$ distribution in Case 3

N		MSE	SE	Bias	L.CI
100	ρ_s	0.331036	0.11940	0.32683	0.33141
	ρ_τ	0.220690	0.07960	0.21788	0.22094
150	ρ_s	0.317019	0.094262	0.32316	0.32617
	ρ_τ	0.211346	0.062841	0.21544	0.217449
200	ρ_s	0.310393	0.080299	0.31867	0.314772
	ρ_τ	0.206929	0.053532	0.21244	0.209848
500	ρ_s	0.307276	0.051039	0.31592	0.20007
	ρ_τ	0.204851	0.034026	0.21061	0.133383
1000	ρ_s	0.300855	0.036114	0.314606	0.141567
	ρ_τ	0.20057	0.024076	0.209737	0.094378

4.2 Analysis of Real Data

The economic data used in this paper is a set of annual observations over 31 years (1980–2010), focusing on two key variables: exports of goods and services (X_1) and GDP growth (X_2). These data were collected from reliable sources, including the World Bank National Accounts and the OECD National Accounts.

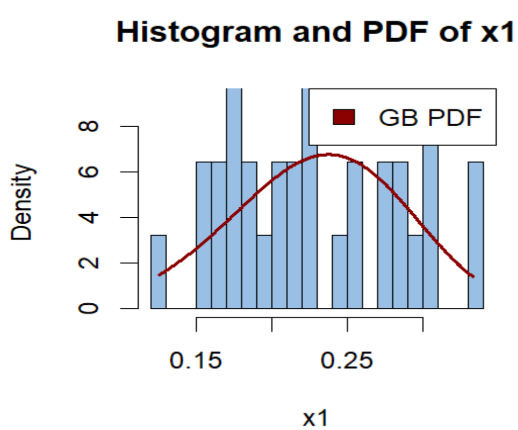
X_1=30.51, 33.37, 27.03, 25.48, 22.35, 19.91, 15.73, 12.56, 17.32, 17.89, 20.05, 27.82, 28.40, 25.84, 22.57, 22.55, 20.75, 18.84, 16.21, 15.05, 16.20, 17.48, 18.32, 21.80, 28.23, 30.34, 29.95, 30.25, 33.04, 24.96, 21.35.

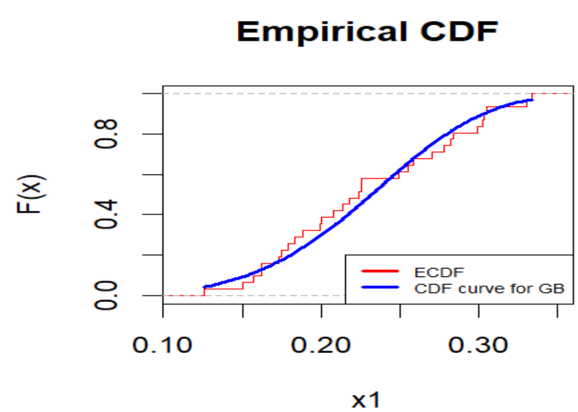
X_2=10.01, 3.76, 9.91, 7.40, 6.09, 6.60, 2.65, 2.52, 7.93, 4.97, 5.70, 1.08, 4.43, 2.90, 3.97, 4.64, 4.99, 5.49, 4.04, 6.11, 5.37, 3.54, 2.37, 3.19, 4.09, 4.48, 6.85, 7.09, 7.16, 4.67, 5.15

Table 7 presents the maximum likelihood estimator (MLE) for the marginal parameters along with multiple goodness-of-fit measures such as the KS test, LL, AIC, BIC, and HQIC. Figure 9 displays the estimated cumulative distribution function (CDF) alongside the empirical CDF and the probability density function (PDF) alongside the histograms. It is clearly evident that the GB model provides a good fit for the marginal data, making the GB model suitable for future analysis of this type of data.

Table *7.* MLE of marginal models: The economic data

		α		ϑ	δ
X_1		4.53456		66.82598	0.1277213
X_2		2.792093		14.86508	0.00490978
X_1					
KS	P-value	LL	AIC	BIC	HQIC
0.12406	0.6809	44.80979	-83.61957	-79.31761	-82.21724
X_2					
KS	P-value	LL	AIC	BIC	HQIC
0.056292	0.999	77.15992	-148.3198	-144.0179	-146.9175





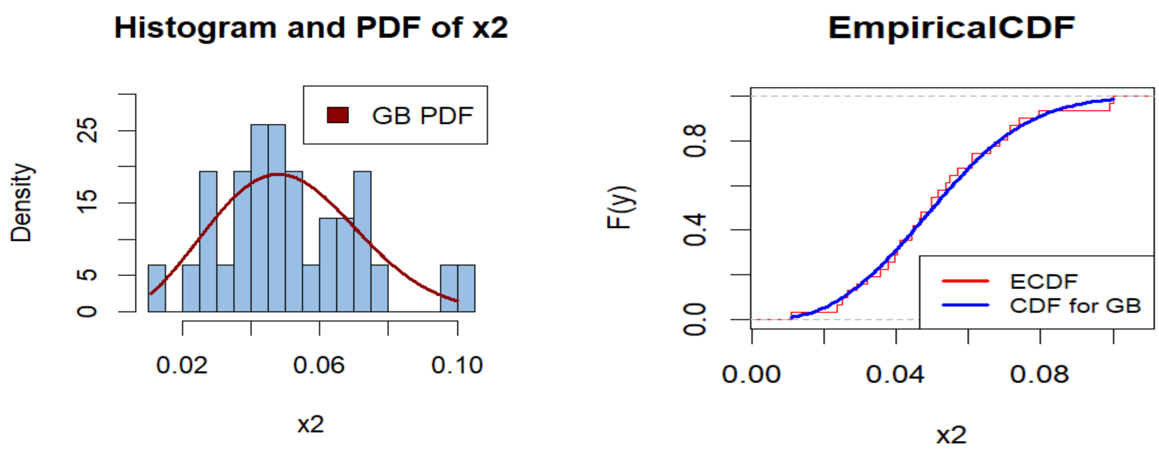


Figure 9. Estimated GB distributions: Economic data

Table 8 presents the maximum likelihood estimates (MLEs) of the parameters for the FGMBGB distribution based on the real economic data for GDP growth and exports of goods and services. These estimates indicate the flexibility and adaptability of the FGMBGB distribution to capture the characteristics of the underlying data. In Table 9, the goodness-of-fit of the FGMBGB model is assessed using the Anderson–Darling (AD) test and various model selection criteria. The AD statistic equals 0.4591 with a p-value (P.V) of 0.2453 (P-value > 0.05), This result suggests that the null hypothesis that the data follows the FGMBGB distribution cannot be rejected, confirming that the model provides a good fit to the economic data additionally, the model selection criteria, LL, AIC, BIC, and HQIC, all exhibit low values, further supporting the adequacy and efficiency of the FGMBGB model in describing the data. Table 10 reports the dependence measures between the two economic variables using the estimated dependence parameter. These values indicate a moderate positive dependence between GDP growth and exports, which is consistent with economic theory, where higher export levels are often associated with increased economic growth. Collectively, these results demonstrate that the FGMBGB distribution is a suitable and reliable model for capturing the marginal behavior and dependence structure in economic data

Table 8. MLE for FGMBGB for economic data.

	$\hat{\alpha}_1$	$\hat{\vartheta}_1$	$\hat{\delta}_1$	$\hat{\alpha}_2$	$\hat{\vartheta}_2$	$\hat{\delta}_2$	$\hat{\theta}$
Estimate	2.547438	2.698615	0.079734	2.3434671	0.75591	0.00057	0.94685

Table 9. Goodness of fit test for economic data for FGMBGB

	AD	P.V	LL	AIC	BIC	HQIC
FGMBGB	0.45906	0.2453	103.8294	-193.6587	-183.6208	-190.3866

Table 10. Dependence measure of FGMBGB for economic data.

	$\hat{\theta}$	$\rho_{sperman}$	$\rho_{kendall}$
FGMBGB	0.94685	0.315618	0.210412

Figure 10 illustrates the probability density function (PDF) of the bivariate FGMBGB distribution between X_1 (exports of goods and services) and X_2 (GDP growth). The contour lines represent different density levels, with the innermost contour 0.2 indicating the region of highest probability density, centered around $X_1 \approx 1$ and $X_2 \approx 0.5$. This suggests that the most likely combination of exports and GDP growth occurs at relatively low values of both variables, potentially indicating a clustering of data points in this region. The spread of the contours outward shows that the probability decreases as we move away from this central region, with a slight elongation along the X_1 axis, hinting at a possible positive correlation between exports and GDP growth.

Figure 11 displays the cumulative distribution function (CDF) of the FGMBGB distribution. The contours represent cumulative probability levels, ranging from 0.1 to 0.9. For instance, the 0.9 contour indicates that 90% of the data points lie below this line, showing the cumulative probability of observing values of X_1 and X_2 up to that point. The contours are more densely packed at lower values of X_1 and X_2, suggesting that a significant portion of the data is concentrated in this region, consistent with the density plot in Figure 10. The gradual spread of the contours toward higher values of X_1 and X_2 further supports a potential positive relationship between the two variables, as higher exports tend to correspond with higher GDP growth in the tail of the distribution.

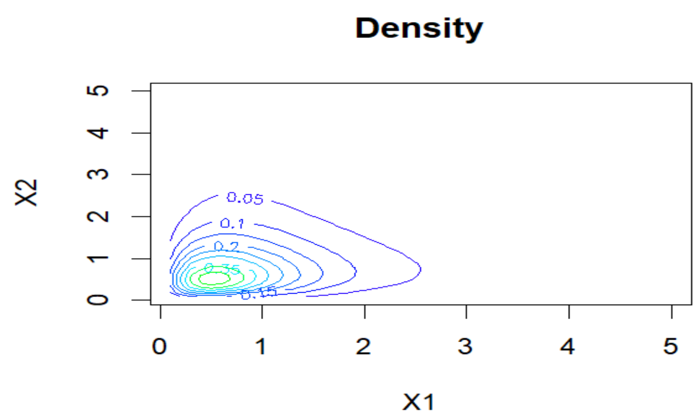


Figure 10. Density contour plot for FGMBGB distribution

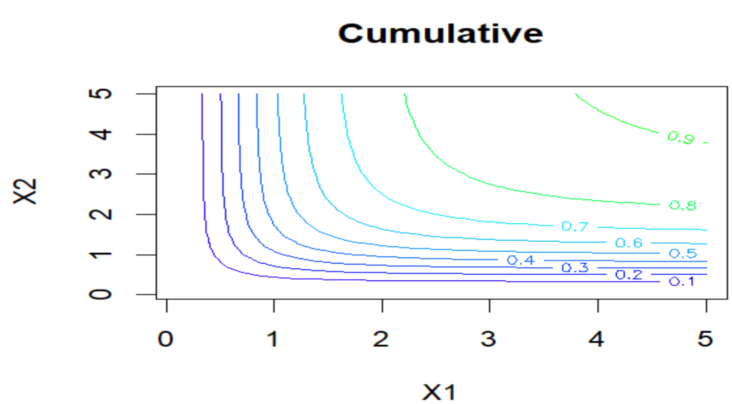


Figure 11. Cumulative contour plot for FGMBGB distribution

5. Conclusion

In this paper, we defined a bivariate GB distribution using the FGM copula function and studied its general structural properties. The model parameters were estimated using Maximum Likelihood Estimation (MLE), and the estimates showed significant stability, approaching the true values as the sample size increased. Tables 1 to 3 show the results obtained from the simulated studies, which include the Bias, SE, MSE, and L.CI for FGMBGB for actual parameters. The result shows, that when n increases, the Bias, SE, MSE, and L.CI decrease, which proves their consistency. Dependence measures such as Kendall's Tau and Spearman's Rho were analyzed to build a model capable of studying relationships and their effects. Tables 4 to 6 show that Kendall's Tau and Spearman's Rho coefficients of FGMBGB distribution of the Bias, SE, MSE, and L.CI decrease as n increases for actual parameters. The model was applied to real economic data, demonstrating its usefulness in analyzing relationships between economic variables, with results showing an improvement in prediction accuracy compared to traditional models. Tables 8 to 10 show the values of MLE, -LL, AIC, BIC, HQIC, and Anderson-Darling (AD) test with its P-value for the fitted FGMBGB model for real data. Finally, the dependence measures between GDP growth and exports indicated a moderate positive dependence between the two variables, consistent with economic theory, which suggests that higher export levels are often associated with greater economic growth. Based on these results, we can conclude that the FGMBGB distribution is a reliable and effective model for analyzing economic data and estimating relationships between economic variables in various practical applications.

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Appendix A

$$I_{11} = -\frac{\partial^2 l(\Theta)}{\partial\, {\alpha_1}^2}\bigg|_{\Theta=\widehat{\Theta}} = \frac{n}{{\alpha_1}^2} + (\vartheta_1 + 1)\Sigma_{i=1}^{n} b(x_{1i};\alpha_1,\vartheta_1,\delta_1) - \Sigma_{i=1}^{n}\varsigma(x_{1i},x_{2i},\Theta)\ ,$$

$$I_{12} = I_{21} = -\frac{\partial^2 l(\Theta)}{\partial\alpha_1\vartheta_1}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n} B(x_{1i},\alpha_1,\delta_1,\vartheta_1) - \Sigma_{i=1}^{n}\varphi(x_{1i},x_{2i},\Theta),$$

$$I_{13} = I_{31} = -\frac{\partial^2 l(\Theta)}{\partial\alpha_1\delta_1}\bigg|_{\Theta=\widehat{\Theta}} = -(\vartheta_1 + 1)\Sigma_{i=1}^{n} \frac{{x_{1i}}^{\alpha_1} log x_{1i}}{(\delta_1 + {x_{1i}}^{\alpha_1})^2} - \Sigma_{i=1}^{n}\Phi(x_{1i},x_{2i},\Theta),$$

$$I_{14} = I_{41} = -\frac{\partial^2 l(\Theta)}{\partial\alpha_1\alpha_2}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\ \varrho(x_{1i},x_{2i},\Theta),$$

$$I_{15} = I_{51} = -\frac{\partial^2 l(\Theta)}{\partial\alpha_1\vartheta_2}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\ \varpi(x_{1i},x_{2i},\Theta),$$

$$I_{16} = I_{61} = -\frac{\partial^2 l(\Theta)}{\partial\alpha_1\delta_2}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\zeta(x_{1i},x_{2i},\Theta),$$

$$I_{17} = -\frac{\partial^2 l(\Theta)}{\partial\alpha_1\theta}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\gamma(x_{1i},x_{2i},\Theta),$$

$$I_{22} = -\frac{\partial^2 l(\Theta)}{\partial\, {\vartheta_1}^2}\bigg|_{\Theta=\widehat{\Theta}} = \frac{n}{{\vartheta_1}^2} - \Sigma_{i=1}^{n}\varepsilon(x_{1i},x_{2i},\Theta),$$

$$I_{23} = I_{32} = -\frac{\partial^2 l(\Theta)}{\partial\vartheta_1\delta_1}\bigg|_{\Theta=\widehat{\Theta}} = \Sigma_{i=1}^{n} - \frac{{x_1}^{\alpha_1}}{{\delta_1}^2(1+\frac{x_1^{\alpha_1}}{\delta_1})} - \xi(x_{1i},x_{2i},\Theta),$$

$$I_{24} = I_{42} = -\frac{\partial^2 l(\Theta)}{\partial\vartheta_1\alpha_2}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\mathrm{A}(x_{1i},x_{2i},\Theta),$$

$$I_{25} = I_{52} = -\frac{\partial^2 l(\Theta)}{\partial\,\vartheta_1\vartheta_5}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\mathcal{g}(x_{1i},x_{2i},\Theta),$$

$$I_{26} = I_{62} = -\frac{\partial^2 l(\Theta)}{\partial\vartheta_1\delta_6}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\mathcal{B}(x_{1i},x_{2i},\Theta)\ ,$$

$$I_{27} = I_{72} = -\frac{\partial^2 l(\Theta)}{\partial\vartheta_1\theta}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\mathbb{N}(x_{1i},x_{2i},\Theta),$$

$$I_{33} = -\frac{\partial^2 l(\Theta)}{\partial\, {\delta_1}^2}\bigg|_{\Theta=\widehat{\Theta}} = \frac{-n}{{\delta_1}^2} + (\vartheta_1 + 1)\Sigma_{i=1}^{n}\left(\frac{-{x_{1i}}^{2\alpha_1}}{{\delta_1}^4(1+\frac{x_{1i}^{\alpha_1}}{\delta_1})^2} + \frac{2{x_{1i}}^{\alpha_1}}{{\delta_1}^3(1+\frac{x_{1i}^{\alpha_1}}{\delta_1})}\right) - \Sigma_{i=1}^{n}\wp(x_{1i},x_{2i},\Theta),$$

$$I_{34} = I_{43} = -\frac{\partial^2 l(\Theta)}{\partial\delta_1\alpha_2}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\mathbb{Q}(x_{1i},x_{2i},\Theta),$$

$$I_{35} = I_{53} = -\frac{\partial^2 l(\Theta)}{\partial\,\delta_1\vartheta_2}\bigg|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n}\mathcal{R}(x_{1i},x_{2i},\Theta),$$

$$I_{36} = I_{63} = -\left.\frac{\partial^2 l(\Theta)}{\partial \delta_1 \delta_2}\right|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n} \Re(x_{1i}, x_{2i}, \Theta),$$

$$I_{37} = I_{73} = -\left.\frac{\partial^2 l(\Theta)}{\partial \delta_1 \theta}\right|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n} \mathbb{R}(x_{1i}, x_{2i}, \Theta),$$

$$I_{44} = -\left.\frac{\partial^2 l(\Theta)}{\partial \alpha_2{}^2}\right|_{\Theta=\widehat{\Theta}} = \frac{n}{\alpha_2{}^2} + (\vartheta_2 + 1)\Sigma_{i=1}^{n} \nu(x_{2i}; \alpha_2, \vartheta_2, \delta_2) - \varsigma(x_{2i}, x_{1i}, \Theta),$$

$$I_{45} = I_{54} = -\left.\frac{\partial^2 l(\Theta)}{\partial \alpha_2 \vartheta_2}\right|_{\Theta=\widehat{\Theta}} = \Sigma_{i=1}^{n} \frac{\log x_{2i}\, x_{2i}{}^{\alpha_2}}{\delta_2(1+\frac{x_{2i}^{\alpha_2}}{\delta_2})} - \Sigma_{i=1}^{n} \mathbb{Z}(x_{1i}, x_{2i}, \Theta),$$

$$I_{46} = I_{64} = -\left.\frac{\partial^2 l(\Theta)}{\partial \alpha_2 \delta_1}\right|_{\Theta=\widehat{\Theta}} = (\vartheta_2 + 1)\Sigma_{i=1}^{n} \left(\frac{\log x_{2i}\, x_{2i}{}^{2\alpha_2}}{\delta_2{}^3(1+\frac{x_{2i}^{\alpha_2}}{\delta_2})^2} - \frac{\log x_{2i}\, x_{2i}{}^{\alpha_2}}{\delta_2{}^2(1+\frac{x_{2i}^{\alpha_2}}{\delta_2})}\right) - \Sigma_{i=1}^{n} \mathcal{F}(x_{1i}, x_{2i}, \Theta),$$

$$I_{47} = I_{74} = -\left.\frac{\partial^2 l(\Theta)}{\partial \alpha_2 \theta}\right|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n} \mathcal{M}(x_{1i}, x_{2i}, \Theta),$$

$$I_{55} = -\left.\frac{\partial^2 l(\Theta)}{\partial \vartheta_2{}^2}\right|_{\Theta=\widehat{\Theta}} = \frac{n}{\vartheta_2{}^2} - \Sigma_{i=1}^{n} \varepsilon(x_{2i}, x_{1i}, \Theta),$$

$$I_{56} = I_{65} = -\left.\frac{\partial^2 l(\Theta)}{\partial \vartheta_2 \delta_2}\right|_{\Theta=\widehat{\Theta}} = \Sigma_{i=1}^{n} - \frac{x_{2i}{}^{\alpha_2}}{\delta_2{}^2(1+\frac{x_{2i}^{\alpha_2}}{\delta_2})} - \Sigma_{i=1}^{n} \hbar(x_{1i}, x_{2i}, \Theta),$$

$$I_{57} = I_{75} = -\left.\frac{\partial^2 l(\Theta)}{\partial \vartheta_2 \theta}\right|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n} ƛ(x_{1i}, x_{2i}, \Theta),$$

$$I_{66} = -\left.\frac{\partial^2 l(\Theta)}{\partial \delta_2{}^2}\right|_{\Theta=\widehat{\Theta}} = \frac{-n}{\delta_2{}^2} + (\vartheta_2 + 1)\Sigma_{i=1}^{n} \left[\frac{-x_{2i}{}^{2\alpha_2}}{\delta_2{}^4(1+\frac{x_{2i}^{\alpha_2}}{\delta_2})^2} + \frac{2x_{2i}{}^{\alpha_2}}{\delta_2{}^3(1+\frac{x_{2i}^{\alpha_2}}{\delta_2})}\right] - \Sigma_{i=1}^{n} \wp(x_{2i}, x_{1i}, \Theta),$$

$$I_{67} = I_{76} = -\left.\frac{\partial^2 l(\Theta)}{\partial \delta_2 \theta}\right|_{\Theta=\widehat{\Theta}} = -\Sigma_{i=1}^{n} \mathbb{C}(x_{1i}, x_{2i}, \Theta),$$

And

$$I_{77} = -\left.\frac{\partial^2 l(\Theta)}{\partial \theta^2}\right|_{\Theta=\widehat{\Theta}} = \Sigma_{i=1}^{n} \mathfrak{H}(x_{1i}, x_{2i}, \Theta).$$

Where

$$b(x_1; \alpha_1, \vartheta_1, \delta_1) = \frac{(\delta_1 + x_1{}^{\alpha_1})\, x_1{}^{\alpha_1} \log[x_1]^2 - \log[x_1]^2 x_1{}^{2\alpha_1}}{(\delta_1 + x_1{}^{\alpha_1})^2}, B(x_1; \alpha_1, \delta_1) = \frac{x_1{}^{\alpha_1} \log[x_1]}{\delta_1 + x_1{}^{\alpha_1}},$$

$$\nu(x_2; \alpha_2, \vartheta_2, \delta_2) = \frac{(\delta_2 + x_2{}^{\alpha_2})\, x_2{}^{\alpha_2} \log[x_2]^2 - \log[x_2]^2 x_2{}^{2\alpha_2}}{(\delta_2 + x_2{}^{\alpha_2})^2},$$

$$\varsigma(x_1, x_2, \Theta) = \{ \frac{-2\theta\vartheta_1 \log[x_1]^2 x_1{}^{\alpha_1} (1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1 - 1} \left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2} - 1\right]}{\delta_1 \left\{1 + \theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1} - 1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2} - 1\right]\right\}} - \{ 2\theta\vartheta_1 \log[x_1]^2 x_1{}^{\alpha_1}$$

$$\cdot[-\frac{log[x_1]x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-2}}{\delta_1}-\frac{\vartheta_1 log[x_1]x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-2}}{\delta_1}]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\}/$$

$$\delta_1\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\} -\frac{1}{\delta_1}2\theta\vartheta_1\log[x_1]\,x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1][-\frac{2\theta\vartheta_1 logx_1x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}}{\delta_1\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta\vartheta_1 logx_1x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_1\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\varphi(x_1,x_2,\Theta)=\{-\frac{2\theta\ log[x_1]x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+$$

$$\frac{2\theta\vartheta_1 log[x_1]\ log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}-\frac{1}{\delta_1}2\theta\vartheta_1\log[x_1]\,x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1][-\frac{2\theta\log(1+\frac{x_1^{\alpha_1}}{\delta_1})(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\cdot\frac{4\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\ (1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\Phi(x_1,x_2,\Theta)=\{\frac{2\theta\vartheta_1\log[x_1]\,x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}$$

$$-\frac{2\theta\vartheta_1\log[x_1]\,x_1^{\alpha_1}\left(\frac{x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-2}}{\delta_1^{\,2}}+\frac{\vartheta_1\,x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-2}}{\delta_1^{\,2}}\right)\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}$$

$$\cdot-\frac{1}{\delta_1}2\theta\vartheta_1\log[x_1]x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]$$

$$\cdot\left[\frac{2\theta\vartheta_1\,x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-\frac{4\theta\vartheta_1\,x_1^{\alpha_1}\ (1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\ (1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}\right]\},$$

$$\varrho(x_1,x_2,\Theta)=\{\frac{4\theta\vartheta_1\vartheta_2 logx_1 logx_2\, x_1{}^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}x_2{}^{\alpha_2}(2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_1\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-$$

$$\frac{1}{\delta_1}2\theta\vartheta_1 log[x_1]x_1{}^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-2\theta\vartheta_2\, log[x_2\,]x_2{}^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta\vartheta_2\, log[x_2\,]\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\, x_2{}^{\alpha_2}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\varpi(x_1,x_2,\Theta)=\{\frac{4\theta\vartheta_1 log[x_1]log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]x_1{}^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_1\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-$$

$$\frac{1}{\delta_1}2\theta\vartheta_1 log[x_1]x_1{}^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-2\theta\, log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right](1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta\, log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\zeta(x_1,x_2,\Theta)=\{-\frac{4\theta\vartheta_2\vartheta_1 logx_1\, x_1{}^{\alpha_1}\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\, x_2{}^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_1\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-$$

$$\frac{1}{\delta_1}2\theta\vartheta_1 logx_1x_1{}^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{2\theta\vartheta_2x_2{}^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-$$

$$\frac{4\theta\vartheta_2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}x_2{}^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\gamma(x_1,x_2,\Theta)=\{-\frac{2\vartheta_1 logx_1\, x_1{}^{\alpha_1}\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-$$

$$\frac{1}{\delta_1}2\theta\vartheta_1 logx_1x_1{}^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-$$

$$\frac{4(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\varepsilon(x_1,x_2,\Theta)=\{\frac{2\theta\log(1+\frac{x_1^{\alpha_1}}{\delta_1})^2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-2\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]$$

$$(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-2\theta\log(1+\frac{x_1^{\alpha_1}}{\delta_1})(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta log\left(1+\frac{x_1^{\alpha_1}}{\delta_1}\right)(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\xi(x_1,x_2,\Theta)=\{\frac{2\theta\, x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{{\delta_1}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-$$

$$\frac{2\theta\,\vartheta_1\log\left(1+\frac{x_1^{\alpha_1}}{\delta_1}\right)x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{{\delta_1}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-2\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}$$

$$\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{2\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}}{{\delta_1}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{{\delta_1}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathrm{A}(x_1,x_2,\Theta)=\{\frac{4\theta\vartheta_2\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]\log[x_2](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-2\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]$$

$$\cdot(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}[-\frac{2\theta\vartheta_2 log x_2\, x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{4\theta\vartheta_2 log x_2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathcal{g}(x_1,x_2,\Theta)=\{\frac{4\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\ (1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_1{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-2\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]$$

$$\cdot(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-2\theta log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]\ (1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathcal{B}(x_1,x_2,\Theta)=\{\frac{-4\theta\,\vartheta_2\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]\ (1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\ x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-2\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]$$

$$\cdot(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{2\theta\vartheta_2\,x_2^{\alpha_2}\ (1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{4\theta\vartheta_2\ (1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\ \ x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathbb{N}(x_1,x_2,\Theta)=\{\frac{-2\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]\ (1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-2\theta\log\left[1+\frac{x_1^{\alpha_1}}{\delta_1}\right]$$

$$\cdot(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_2{}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-$$

$$\frac{4\ (1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\ (1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\wp(x_1,x_2,\Theta)=\{\frac{-4\ \theta\ \vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1{}^3\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+$$

$$\frac{2\theta\vartheta_1 x_1^{\alpha_1}\left(\frac{x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-2}}{\delta_1^{\,2}}+\frac{\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-2}}{\delta_1^{\,2}}\right)\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+\frac{1}{\delta_1^{\,2}}2\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\left[\frac{2\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-\right.$$

$$\left.\frac{4\theta\vartheta_1\, x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}\right]\},$$

$$\mathbb{Q}(x_{1i},x_{2i},\Theta)=\{\frac{-4\,\theta\,\vartheta_1\vartheta_2\log x_2\, x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}x_2^{\alpha_2}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{\delta_1^{\,2}\,\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+\frac{1}{\delta_1^{\,2}}2\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\left[\frac{-2\theta\vartheta_2 x_2^{\alpha_2}\log x_2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\right.$$

$$\left.\frac{4\theta\vartheta_2\log[x_2](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}\right]\},$$

$$\mathcal{R}(x_1,x_2,\Theta)=\{\frac{-4\,\theta\vartheta_1\log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+\frac{1}{\delta_1^{\,2}}2\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\left[\frac{-2\theta\log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right](1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{4\theta\log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right](1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}\right]\}$$

$$\Re(x_1,x_2,\Theta)=\{\frac{4\,\theta\vartheta_1\vartheta_2 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_1^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+\frac{1}{\delta_1^{\,2}}2\theta\vartheta_1 x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\left[\frac{2\theta\vartheta_2 x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-\right.$$

$$\left.\frac{4\theta\vartheta_2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2^{\,2}\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}\right]\},$$

$$\mathbb{R}(x_1,x_2,\Theta)=\frac{2\,\theta\vartheta_1 x_1^{\alpha_1}\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]}{{\delta_1}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+\frac{1}{{\delta_1}^2}2\theta\vartheta_1\,x_1^{\alpha_1}(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1-1}$$

$$\cdot\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right][\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-$$

$$\frac{4\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathbb{Z}(x_1,x_2,\Theta)=\{\frac{-2\theta\log[x_2]\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+$$

$$\frac{2\theta\vartheta_2\log[x_2]\log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-\frac{1}{\delta_2}2\theta\vartheta_2\;\log x_2\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]$$

$$\cdot\;x_2^{\alpha_2}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}[\frac{-2\,\theta\,\log(1+\frac{x_2^{\alpha_2}}{\delta_2})(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{4\,\theta\,\log(1+\frac{x_2^{\alpha_2}}{\delta_2})(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathcal{F}(x_1,x_2,\Theta)=\frac{2\theta\,\vartheta_2\log[x_2]\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+$$

$$\frac{2\theta\vartheta_2\log[x_2]\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}\left[\frac{x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-2}}{{\delta_2}^2}+\frac{\vartheta_2\,x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-2}}{{\delta_2}^2}\right]}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}$$

$$-\frac{1}{\delta_2}2\theta\vartheta_2\log x_2\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}$$

$$\cdot[\frac{2\,\theta\,\vartheta_2\,x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-\frac{4\,\theta\vartheta_2\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\;x_2^{\alpha_2}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathcal{M}(x_1,x_2,\Theta)=\{\frac{-2\,\vartheta_2\log x_2\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{\delta_2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-\frac{1}{\delta_2}2\theta\vartheta_2\log x_2\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]$$

$$\cdot\, x_2^{\alpha_2}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}[\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{2\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-$$

$$\frac{4\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{(1+\theta(2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1)(2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1))^2}]\},$$

$$\hbar(x_1,x_2,\Theta)=\{\frac{2\,\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-$$

$$\frac{2\theta\vartheta_2\log[1+\frac{x_2^{\alpha_2}}{\delta_2}]\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-2\theta log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right](1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}$$

$$[\frac{2\,\theta\,\vartheta_2 x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}-\frac{4\,\theta\,\vartheta_2 x_2^{\alpha_2}\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\lambda(x_1,x_2,\Theta)=\{\frac{-2\log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}-2\theta\log\left[1+\frac{x_2^{\alpha_2}}{\delta_2}\right]$$

$$\cdot\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right](1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}[\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{2\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}$$

$$-\frac{4\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathbb{C}(x_1,x_2,\Theta)=\{\frac{2\,\vartheta_2\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}}{{\delta_2}^2\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}}+\frac{1}{{\delta_2}^2}2\theta\vartheta_2\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]x_2^{\alpha_2}$$

$$\cdot\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2-1}[\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+$$

$$\frac{2\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}$$

$$-\frac{4\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\},$$

$$\mathfrak{H}(x_1,x_2,\Theta)=\left\{\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right][\frac{-1}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\right.$$

$$\frac{2\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}+\frac{2\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}$$

$$-\frac{4\,(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}\,(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}}{\left\{1+\theta\left[2(1+\frac{x_1^{\alpha_1}}{\delta_1})^{-\vartheta_1}-1\right]\left[2(1+\frac{x_2^{\alpha_2}}{\delta_2})^{-\vartheta_2}-1\right]\right\}^2}]\}.$$