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Inference for a Simple Step-Stress Model for the Extension of Exponential Distribution under Progressive Type-II Censored Competing Risks Data

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Accelerated life testing; Progressive type-II censoring; Competing risks; Extension of the exponential distribution; Cumulative exposure model; Maximum likelihood estimation; Bayesian estimation.

Abstract

In a special class of accelerated life tests known as step stress tests, the stress levels increase discretely at pre-fixed time points, and this allows the experimenter to obtain information on the parameters of the lifetime distributions more quickly than under normal operating conditions. In this paper, we consider the simple step-stress model with competing risks under progressive Type-II censoring. The cumulative exposure model is assumed when the lifetime of test units follows an extension of the exponential Nadarajah-Haghighi distribution. Under this setup, we obtain the maximum likelihood estimates and the Bayes Estimators of the unknown parameters using the Markov chain Monte Carlo method under various loss functions. Also, the confidence intervals are derived by using the asymptotic distributions of the maximum likelihood estimates, and we obtained the highest posterior density credible intervals based on different prior distributions. Their performance is assessed through Monte Carlo simulations, and finally, we illustrate the methods of inference discussed here with an example.

Mathematical Subject Classification: 62F10, 62F15, 62F25

1. Introduction

In reliability analysis, it's common to consider several causes, either mechanical or electrical, those competing to fail a unit. These causes are called "competing risks." The main problem of reliability studying or life testing experiments is that units' failure connects with more than one cause of failure. In practice, the failure causes may be independent or dependent.

Although a dependent risk structure might be more realistic, there is a concern about the identifiability of the underlying model. Without the information on covariates, it's not possible to test the assumption of s-independent risks, as discussed by Crowder (1991) and Kalbfleisch and Prentice (2011). Thus, in most

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cases, for analyzing a competing risks model, the causes of failure are assumed to be independent. A series system with multiple components can also be treated as an independent competing risks model. By assuming independence among competing failure causes, many authors have investigated the competing risks models, such as David and Moesch Berger (1978), Sarhan et al. (2010), and Crowder (2001). Censoring is common in life tests for saving time and cost reduction. There are different types of censoring schemes. The most popular censoring schemes are Type-I and Type-II censoring schemes. In the Type-I censoring scheme, the test is terminated at fixed $T \in (0, \infty)$ and uses the observed samples until T. In the Type-II censoring scheme, the test is terminated when samples are observed until the predetermined observation number (r). Different types of mixtures of these two basic censoring schemes are known as hybrid censoring schemes. It was first introduced by Epstein (1954), in the state of life-testing Experiments.

The hybrid censoring schemes have been studied under different lifetime distributions, by several authors. Compared to the conventional Type-I and Type-II censoring schemes, progressive censoring provides higher flexibility to the experimenter by allowing the tested units to be removed at non-terminal time points. This allowance may be desirable when some degradation-related information from live test specimens needs to be collected and analyzed, or when scarce testing facilities need to be released for other experiments. So progressive Type-II censoring is highly efficient and effective in utilizing the available resources. Although censoring can help shorten the time and reduce the cost, with more and more products possessing high quality and long life, censoring can't meet the demands of collecting enough information about the products' lifetimes as soon as possible. One way to overcome this problem is accelerated life tests (ALTs) [see Nelson (1990)]. So, the ALTs are widely used in reliability analysis. Shi et al. (2013) considered a constant stress accelerated life test (CSALT) with competing risks for failure from exponential distribution under progressive Type II hybrid censoring. The maximum likelihood estimator and Bayes estimator of the parameter were derived. Xu and Tang (2011) and Wu et al. (2014) considered different inferential issues regarding the constant-stress accelerated competing failure models when the lifetime of different risk factors follows Weibull distributions. Based on Nelson's cumulative exposure (CE) model.

Balakrishnan and Han (2008) and Han and Balakrishnan (2010) developed the exact inference for a simple step-stress model with competing risks for failure from the exponential distribution under Type-II and Type-I censoring schemes, respectively. Liu and Shi (2017) considered a simple step-stress model with progressively censored competing risk data from the Weibull distribution. El-Shahat et al. (2025) considered a simple step-stress model with Type-I censored competing risks data from Rayleigh distribution. Also, Han and Kundu (2014) introduced a step-stress model with competing risks for failure from the generalized exponential distribution under Type-I censoring. Recently, the progressive Type-II censoring scheme has become quite popular for analyzing highly reliable data.

2. Progressive Type-II Censoring Scheme

Progressive Type-II censoring was introduced by Cohen (1963). This type of censoring scheme can be described as: Suppose n identical items are put to test; the integer $m \leq n$ is a pre-specified number of failures and $R_1, R_2, \dots, R_m$ are m prefixed integers satisfying $R_1 + R_2 + \cdots . + R_m + m = n$. At the time of the first failure $t_{1:m:n}, R_1$ of the surviving units are randomly withdrawn. Likewise, at the time of the second failure $t_{2:m:n}, R_2$ of the surviving units are randomly withdrawn, and so on. At the time of the ith failure $t_{i:m:n}$, the experiment stopped and all surviving $R_i = n - m - \left(R_1 + \cdots . + R_{(m-1)}\right)$ units are withdrawn. For more details about progressive Type-II censoring, the experiment is stopped, and all surviving [Equation]units are withdrawn. For more details about progressive Type-II censoring, Balakrishnan and Aggarwala. (2000), and Balakrishnan and Cramer. (2014).

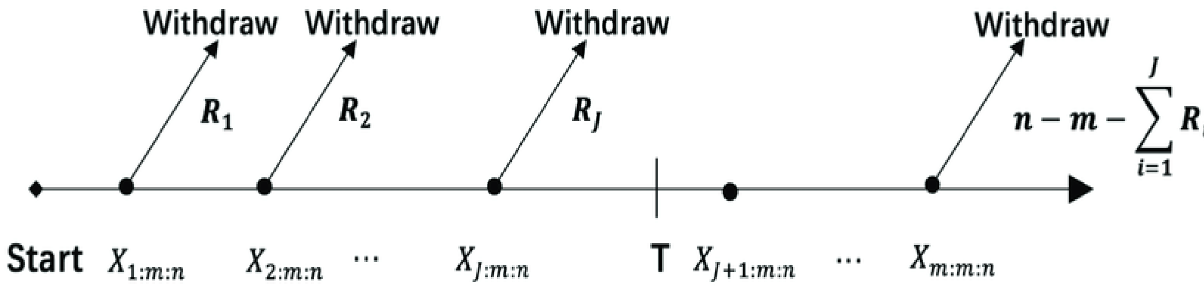

Figure 1. Progressive Type-II Censoring Scheme

Nadarajah and Haghighi (2011) introduced an extension of the exponential distribution as an alternative to gamma, Weibull, and generalized exponential distributions. It has an increasing as well as a decreasing failure rate depending on the values of the shape parameter. It has an increasing (decreasing) failure rate function when $(\alpha > 1)$, $(\alpha < 1)$, it respectively, and for $\alpha = 1$ becomes constant. The extension of the exponential distribution specified by the probability density function (PDF):

$$f(t) = \alpha\,\beta\,(1+\beta t)^{\alpha-1}\,e^{[1-(1+\beta t)^{\alpha}]} \quad ,t > 0\,,\ \alpha,\beta > 0.$$

The corresponding cumulative distribution function (CDF) is

$$F(t) = 1 - e^{[1-(1+\beta t)^{\alpha}]}, t > 0\,, \alpha,\beta > 0 \tag{1}$$

and the corresponding hazard rate function (HRF) is given by:

$$h(t) = \alpha\,\beta\,(1\ \beta t)^{\alpha-1} \tag{2}$$

where β is the scale parameter, and α is the shape parameter.

This research is organized as follows; Section 3 introduces the model formulation. The classical maximum likelihood estimation (MLEs) and the asymptotic confidence intervals (CIs) of the unknown parameters are discussed in Section 4. In Section 5, The Bayes estimates (BEs) of model parameters using the Markov chain Monte Carlo (MCMC) method are obtained. In Section 6, the performance of these confidence/credible intervals is evaluated in terms of probability coverages via Monte Carlo simulations. In Section 7, we present a numerical example to illustrate all the methods of inference developed in this article, and some concluding remarks are finally made in Section 8.

3. Model description and test assumptions

Let s_0 denote the normal stress level. Consider n identical units being placed on a simple step-stress ALT under the initial stress level $s_1(s_1. > s_0)$ See Chapter 2 of Nelson (1990) for further information on accelerated models.

The successive failure times are recorded along with the information about which risk factor causes each failure. At a pre-fixed time $\tau \in (0,\infty)$, the stress level is increased from $s_1 to\ s_2$ and the life test continues until the $mth\,(m$ is pre-fixed) failure is observed. At the $(1 \le i \le m-1)$ failure time, R_i of the surviving items are randomly removed, where R_i is pre-fixed and $R_m = n - m - \sum_{i=1}^{m-1} R_i$.

Basic Assumptions

1. The failure of a product occurs only due to one of the two independent competing failures caused by lifetimes. T_1 and T_2. Then, the failure time of the product $T = min(T_1, T_2)$.
2. Two stress levels $s_1 and\ s_2 (s_1 < s_2)$ are used (simple step stress).
3. For any stress level s_i , the lifetime of the jth failure cause T_{ij} , $(i, j = 1,2)$ follows an extension of the exponential distribution with a scale parameter $\beta_{ij}$and shape parameter α_{ij}.
4. The failure mechanisms are the same under different stress levels, i.e. $\alpha_{1j} = \alpha_{2j} = \alpha_j$
5. The accelerated function (AF) of the jth failure cause is log-linear, namely.
$$ln(\beta_{ij}) = a_j + b_j\ \varphi(s_i)\ , i, j = 1,2\ , \tag{3}$$
where a_j, b_j are unknown parameters, $\varphi(s_i)$ is a given decreasing function of stress level s. We adopt the Arrhenius model in this article, so $\varphi(s_i) = 1/s_i$.
6. The cumulative exposure model holds, see (Nelson, 1990).From the assumption of the cumulative exposure model and the CDF given in Eq. (1), the CDF of the lifetime T_{ij} under the simple step-stress ALT are given as

$$G_j(t) = \begin{cases} F_1(t) = 1 - e^{\left[1-(1+\beta_{1j}t)^{\alpha_j}\right]} & if\ \ 0 < t < \tau \\ F_2(t - \tau + u) = 1 - e^{\left[1-(1+\beta_{2j}(t-\tau)+\beta_{1j}\tau)^{\alpha_j}\right]} & if\ \ \tau \le t < \infty \end{cases} \tag{4}$$

Where the equivalent starting time u is the solution of $F_2(u) = F_1(\tau)$ for j=1, 2. The corresponding probability density function (PDF) of T_j is given by

$$g_j(t) = \begin{cases} f_1(t) = \alpha_j\,\beta_{1j}\,(1 + \beta_{1j})^{\alpha_j - 1}\ e^{\left[1-(1+\beta_{1j}t)^{\alpha_j}\right]} & if\ 0 < t < \tau \\ f_2(t - \tau + u) = \alpha_j\,\beta_{2j}\,(1 + \beta_{2j}(t - \tau) + \beta_{1j}\tau)^{\alpha_j} e^{\left[1-(1+\beta_{2j}(t-\tau)+\beta_{1j}\tau)^{\alpha_j - 1}\right]} & if\ \tau \le t < \infty \end{cases} \tag{5}$$

for j = 1, 2.

Since we will observe only the smaller of T_1 and T_2, let T = min (T_1, T_2) note the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$F_T(t) = 1 - (1 - G_1(t))(1 - G_2(t))$$

$$F_T(t) = \begin{cases} 1 - \{e^{[1-(1+\beta_{11}t)^{\alpha_1}]}\}\{e^{[1-(1+\beta_{12}t)^{\alpha_2}]}\} & if\ \ 0 < t < \tau \\ 1 - \{e^{[1-(1+\beta_{21}(t-\tau)+\beta_{11}\tau)^{\alpha_1}]}\}\{e^{[1-(1+\beta_{22}(t-\tau)+\beta_{12}\tau)^{\alpha_2}]}\} & if\ \ \tau \le t < \infty \end{cases} \tag{6}$$

$$f_T(t) = \begin{cases} \alpha_1\beta_{11}(1 + \beta_{11}t)^{\alpha_1 - 1} + \alpha_2\beta_{12}(1 + \beta_{12}t)^{\alpha_2 - 1}\ e^{[1-(1+\beta_{11}t)^{\alpha_1}]+[1-(1+\beta_{12}t)^{\alpha_2}]} & if\ 0 < t < \tau \\ \alpha_1\beta_{21}(1 + \beta_{21}(t - \tau) + \beta_{11}\tau)^{\alpha_1 - 1} + \alpha_2\beta_{22}(1 + \beta_{22}(t - \tau) + \beta_{12}\tau)^{\alpha_2 - 1} & \\ \quad \times\ e^{[1-(1+\beta_{21}(t-\tau)+\beta_{11}\tau)^{\alpha_1}]+[1-(1+\beta_{22}(t-\tau)+\beta_{12}\tau)^{\alpha_2}]} & if\ \ \tau \le t < \infty \end{cases} \tag{7}$$

Let δ be the indicator of the failure cause, then we derive the joint PDF of (T, δ) as

$$f_{T,\delta}(t, j) = g_j(t)\left(1 - G_{j`}(t)\right)$$

$$f_{T,\delta}(t,j) = \begin{cases} \alpha_j\beta_{1j}\left(1+\beta_{1j}t\right)^{\alpha_j-1}\ e^{[1-(1+\beta_{11}t)^{\alpha_1}]+[1-(1+\beta_{12}t)^{\alpha_2}]} & if\ \ 0 < t < \tau \\ \alpha_j\beta_{2j}\left(1+\beta_{2j}(t-\tau)+\beta_{1j}\tau\right)^{\alpha_j-1} e^{[1-(1+\beta_{21}(t-\tau)+\beta_{11}\tau)^{\alpha_1}]+[1-(1+\beta_{22}(t-\tau)+\beta_{12}\tau)^{\alpha_2}]} & if\ \tau \le t < \infty \end{cases} \tag{8}$$

for $j, j` = 1, 2$ and $j \neq j`$.

Suppose there are N_1 failures before the stress changing time τ. If we denote n_{1j} and n_{2j} $(j = 1,2)$ as the number of failures due to failure causes j under stress level $s_1 and$ s_2 , respectively, then $N_1 = n_{11} + n_{12}$ and $m - N_1 = n_{21} + n_{22}$ is the number of failures under stress level s_2 . Since each failure time is accompanied by the corresponding cause of failure. Let $\delta = (\delta_1, \delta\{_2, \ldots, \delta\{_m\}$ be the observed sequence of the cause of failure corresponding to the observed failure time $t = (t_1, t_2, \ldots, t_m)$.

4. Maximum Likelihood Estimates

Under the assumption of the cumulative exposure model, we formulate the likelihood function of $\Theta = \left(\alpha_j, \beta_{1j}, \beta_{2j}\right)$ based on the progressive Type-II censored data as

$$L \propto \prod_{i=1}^{N_1} f_{T,\delta}(t_i, \delta_i)\,[1 - F(t_i)] \prod_{i=N_1+1}^{m} f_{T,\delta}(t_i, \delta_i)\ \ [1 - F(t_i)]^{R_i} \tag{9}$$

$$\begin{aligned} L \propto & \prod_{i,j=1}^{2} \alpha_j^{n_{ij}}\, \beta_{ij}^{n_{ij}}\, \prod_{i=1}^{N_1}\left(1+\beta_{1j}t_i\right)^{\alpha_j-1}\ e^{(R_i+1)\left[1-\left(1+\beta_{1j}t_i\right)^{\alpha_j}\right]} \\ & \times\ \prod_{i=N_1+1}^{m}\left(1+\omega(t_i)\right)^{\alpha_j-1} e^{(R_i+1)\left[1-\left(1+\omega(t_i)\right)^{\alpha_j}\right]} \end{aligned} \tag{10}$$

where, $\omega(t_i) = \{\beta_{2j}(t_i - \tau) + \beta_{1j}\tau\}$

Then the logarithm of likelihood function Eq. (10) of α_j, β_{1j} and β_{2j} is given by:

$$\begin{aligned} lnL \propto\ & n_{ij}ln\alpha_j + n_{1j}ln\beta_{ij} + n_{2j}ln\beta_{2j} + \left(\alpha_j - 1\right)\sum_{i=1}^{N_1} ln\left(1+\beta_{1j}t_i\right) \\ & + \sum_{i=1}^{N_1}(R_i+1)\left[1-\left(1+\beta_{1j}t_i\right)^{\alpha_j}\right] + \left(\alpha_j - 1\right)\sum_{i=N_1+1}^{m} ln\left(1+\omega(t_i)\right) \\ & + \sum_{i=N_1+1}^{m}(R_i+1)\left[1-\left(1+\omega(t_i)\right)^{\alpha_j}\right] \end{aligned} \tag{11}$$

4.1 Point Estimation

The MLEs of the parameters α_j, β_{1j} and β_{2j} can be obtained by setting the first partial derivative of Eq. (11) about $\alpha_j, \beta_{1j} and \beta_{2j}$ to zero, namely

$$\begin{aligned} \frac{\partial lnL}{\partial\alpha_j} = & \frac{\sum_{i=1}^{2} n_{ij}}{\alpha_j} + \sum_{i=1}^{N_1} ln\left(1+\beta_{1j}t_i\right) - \sum_{i=1}^{N_1}(R_i+1)\left(1+\beta_{1j}t_i\right)^{\alpha_j}\ ln\left(1+\beta_{1j}t_i\right) \\ & + \sum_{i=N_1+1}^{m} ln\left(1+\omega(t_i)\right)\ - \sum_{i=N_1+1}^{m}(R_i+1)\left(1+\omega(t_i)\right)^{\alpha_j} ln\left(1+\omega(t_i)\right) \end{aligned} \tag{12}$$

$$\frac{\partial lnL}{\partial \beta_{1j}} = \frac{n_{1j}}{\beta_{1j}} + (\alpha_j - 1)\sum_{i=1}^{N_1}\frac{t_i}{(1+\beta_{1j}t_i)} - \alpha_j\sum_{i=1}^{N_1}(R_i+1)\ t_i\ (1+\beta_{1j}t)^{\alpha_j-1}$$
$$+(\alpha_j - 1)\sum_{i=N_1+1}^{m}\frac{\tau}{(1+\omega(t_i))} - \alpha_j\sum_{i=N_1+1}^{m}(R_i+1)\ \tau\ (1+\omega(t_i))^{(\alpha_j-1)} \tag{13}$$

$$\frac{\partial lnL}{\partial \beta_{2j}} = \frac{n_{2j}}{\beta_{2j}} + \sum_{i=N_1+1}^{m}\frac{(t_i-\tau)}{(1+\omega(t_i))} - \alpha_j\sum_{i=N_1+1}^{m}(R_i+1)\ (t_i-\tau)\ (1+\omega(t_i))^{(\alpha_j-1)} \tag{14}$$

Now, we have a system of six nonlinear equations in six unknowns α_j, β_{1j} and β_{2j} a closed-form solution is quite difficult to obtain. Consequently, an iterative procedure such as the Newton-Raphson algorithm can be used to obtain a numerical solution of the above nonlinear system.

4.2 Asymptotic Confidence Intervals

The asymptotic variance and covariance matrix of maximum likelihood estimates is given by the elements of the inverse of the Fisher information matrix as follows.

$$I_{ij}(\Theta) = E\{-\frac{\partial^2 lnL}{\partial\theta_i\,\partial\theta_j}\}$$

However, the exact mathematical expressions for the previous expectation are often difficult to obtain (Cohen, 1965). Therefore, the Fisher information matrix is approximated as

$$I_{ij}(\Theta) \approx \{-\frac{\partial^2 lnL}{\partial\theta_i\,\partial\theta_j}\}$$

The asymptotic variance-covariance matrix is defined by inverting the Fisher information matrix $I_{ij}^{-1}(\Theta)$ as follows:

$$I^{-1}(\alpha_j, \beta_{1j}, \beta_{2j}) = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}^{-1}_{\alpha_j,\beta_{ij}=\widehat{\alpha_j},\widehat{\beta_{ij}}}$$

here,

$$L_{11} = \frac{\partial^2 lnL}{\alpha_j^2} = -\frac{\sum_{i=1}^{2} n_{ij}}{\alpha_j^2} - \sum_{i=1}^{n_1}(R_i+1)(1+\beta_{1j}t_i)^{\alpha_j}\ [ln(1+\beta_{1j}t_i)]^2$$
$$-\sum_{i=n_1+1}^{m}(R_i+1)(1+\omega(t_i))^{\alpha_j}\ ln(1+\omega(t_i))^2,$$

$$L_{12} = L_{21} = \frac{\partial^2 lnL}{\partial\alpha_j\beta_{1j}} = \sum_{i=1}^{n_1}\frac{t_i}{(1+\beta_{1j}t_i)} - \sum_{i=1}^{n_1}(R_i+1)\ t_i\ (1+\beta_{1j}t_i)^{\alpha_j-1}[1+\alpha_j ln(1+\beta_{1j}t_i)]$$
$$+\sum_{i=n_1+1}^{m}\frac{\tau}{(1+\omega(t_i))} - \sum_{i=n_1+1}^{m}\tau(R_i+1)(1+\omega(t_i))^{\alpha_j-1}[1+\alpha_j ln(1+\omega(t_i))],$$

$$L_{13} = L_{31} = \frac{\partial^2 logL}{\partial\alpha_j\beta_{2j}}$$

$$= \sum_{i=n_1+1}^{m} \frac{(t_i-\tau)}{\left(1+\omega(t_i)\right)} - \sum_{i=n_1+1}^{m} (t_i-\tau)(R_i+1)\left(1+\omega(t_i)\right)^{\alpha_j-1}\left[1+\alpha_j ln\left(1+\omega(t_i)\right)\right],$$

$$L_{22} = \frac{\partial^2 logL}{\beta_{1j}^2} = -\frac{n_{1j}}{\beta_{1j}^2} - (\alpha_j-1)\sum_{i=1}^{n_1} \frac{t_i^2}{\left(1+\beta_{1j}t_i\right)^2} - \alpha_j(\alpha_j-1)\sum_{i=1}^{n_1}(R_i+1)\ t_i^2\ \left(1+\beta_{1j}t_i\right)^{\alpha_j-2}$$
$$-(\alpha_j-1)\sum_{i=n_1+1}^{m} \frac{\tau^2}{\left(1+\omega(t_i)\right)^2} - \alpha_j(\alpha_j-1)\sum_{i=n_1+1}^{m}(R_i+1)\ \tau^2\ \left(1+\omega(t_i)\right)^{\alpha_j-2},$$

$$L_{23} = L_{32} = \frac{\partial^2 logL}{\partial\beta_{1j}\beta_{2j}}$$
$$= -(\alpha_j-1)\sum_{i=n_1+1}^{m} \frac{\tau(t_i-\tau)}{\left(1+\omega(t_i)\right)^2} - \alpha_j(\alpha_j-1)\sum_{i=n_1+1}^{m}(R_i+1)\ \tau(t_i-\tau)\ \left(1+\omega(t_i)\right)^{\alpha_j-2},$$

$$L_{33} = \frac{\partial^2 logL}{\beta_{2j}^2} = -\frac{n_{2j}}{\beta_{2j}^2} - (\alpha_j-1)\sum_{i=n_1+1}^{m} \frac{(t_i-\tau)^2}{\left(1+\omega(t_i)\right)^2}$$
$$-\alpha_j(\alpha_j-1)\sum_{i=n_1+1}^{m}(R_i+1)\ (t_i-\tau)^2\ \left(1+\omega(t_i)\right)^{\alpha_j-2} \tag{15}$$

Substitute the MLEs $\hat{\alpha}_j, \hat{\beta}_{1j}$ and $\hat{\beta}_{2j}$ for α_j, β_{1j} and β_{2j}, the observed Fisher information matrix $I_{ij}(\Theta)$ can be obtained. Upon inverting this matrix and denoting $\hat{V}_{ij} = \hat{I}_{ij}^{-1}$we derive the two-sided) $100(1-\gamma)$ ACIs for $\alpha_j, \beta_{1j} and \beta_{2j}$ as

$$\widehat{\alpha_j} \pm z_{\gamma/2}\sqrt{\hat{V}_{ij}(1,1)}\,,\ \widehat{\beta_{1j}} \pm z_{\gamma/2}\sqrt{\hat{V}_{ij}(2,2)}\,,\ \widehat{\beta_{2j}} \pm z_{\gamma/2}\sqrt{\hat{V}_{ij}(3,3)}\ .$$

respectively, where $j = 1,2 and z_{\gamma/2}$ is the $(1 - z\_\gamma/2\}\ )th$ quantile of a standard normal distribution.

5. Bayes Estimation

In this section, we consider the Bayesian estimates of unknown parameters. Bayesian estimation approach has received a lot of attention for ana
lyzing failure time data. It makes use of one's prior knowledge about the parameters and also takes into consideration the data available.

5.1 Point Estimation

The square error (SE) loss function and linear exponential (LINEX) loss function are considered to obtain BEs of the model parameters $\alpha_j, \beta_{1j} and\ \beta_{2j}$ under progressive Type-II censoring.

- **Under Squared Error Loss Function (SE)**

The Bayes estimates of the unknown parameters $\theta = \alpha_j, \beta_{1j}$ and β_{2j}) , $j = 1,2$ under (SE) denoted by $\tilde{\theta}_{(BSE)}$; can be calculated through the following equations as follows

$$\tilde{\theta}_{(BSE)} = E\left(\theta\middle|\underline{x}\right) = \int_0^\infty \theta\pi^*\left(\theta\middle|\underline{x}\right)\mathrm{d}$$

- **Under LINEX Loss Function**

Under the assumption that the minimal loss occurs at $\theta = \left(\alpha_j, \beta_{1j}\beta_{2j}\right), j = 1,2$, the LINEX loss function can be expressed as

$$L_{BL}(\Delta) \propto e^{c\Delta} - c\Delta - 1\,, c \neq 1 \tag{16}$$

where $\Delta = \left(\hat{\theta} - \theta\right)$ is an estimate of θ.

The posterior expectation of the LINEX loss function Eq. (16) is

$$E\left(L_{BL}\left(\hat{\theta} - \theta\right)\right) \propto e^{c\hat{\theta}}\, E\left[e^{c\hat{\theta}}\right] - c\left(\hat{\theta} - E[\theta]\right) - 1 \tag{17}$$

The Bayes estimator of θ, denoted by $\hat{\theta}$ under LINEX loss function, is the value of $\hat{\theta}$ which minimizes Eq. (17) .it is

$$\tilde{\theta}_{BL} = -\frac{1}{c} log\,\{E\left[e^{c\hat{\theta}}\right]\} \tag{18}$$

For Bayesian estimation, we need prior distribution for the parameters α_j, β_{1j} and β_{2j} .Therefore, we suppose the prior distribution of α_j is non-informative prior

$$\pi\left(\alpha_j\right) = \frac{1}{\alpha_j}\,,\ \alpha_j > 0.$$

We Assume that the model parameters β_{1j} and β_{2j} are independent with priors as:

$$\pi\left(\beta_{1j}\right) \propto \beta_{1j}^{\sigma_{1j}-1}\, e^{\lambda_{1j}\beta_{1j}}\ ,\ \sigma_{1j}\,, \lambda_{1j}\quad .$$

and

$$\pi\left(\beta_{2j}\right) \propto \beta_{2j}^{\sigma_{2j}-1}\, e^{\lambda_{2j}\beta_{2j}}\ ,\ \sigma_{2j}\,, \lambda_{2j}$$

Respectively, many authors used the gamma prior for the scale and the shape parameters instead of the exponential prior because the gamma prior is wealthy enough to cover the prior belief of the experimenter. See (Nassar and Eiss, 2005) for Weibull distribution and Singh et al. (2014) for the extension of the exponential distribution. Thus, the joint prior PDF of α_j, β_{1j} and β_{2j} is given by:

$$\pi\left(\alpha_j, \beta_{1j}\beta_{2j}\right) \propto \alpha_j^{-1}\beta_{1j}^{\sigma_{1j}-1}\ \beta_{2j}^{\sigma_{2j}-1} e^{-\left(\lambda_{1j}\beta_{1j}+\lambda_{2j}\beta_{2j}\right)}\,,\ \alpha_j, \beta_{1j}\beta_{2j}\ > 0 \tag{19}$$

The joint posterior density function of the parameters α_j, β_{1j} and β_{2j} can be written from Eq. (10) and Eq. (19)

$$\begin{aligned}\pi^*\left(\alpha_j, \beta_{1j}\beta_{2j}\right) = \frac{1}{\nu}\,\{&\alpha_j^{n_{ij}-1}\beta_{1j}^{n_{1j}+\sigma_{1j}-1}\beta_{2j}^{n_{2j}+\sigma_{2j}-1}\ e^{-\left(\lambda_{1j}\beta_{1j}+\lambda_{2j}\beta_{2j}\right)}\\ &\times \prod_{i=1}^{N_1}\left(1 + \beta_{1j}t_i\right)^{\alpha_j-1}\ e^{(R_i+1)\left[1-\left(1+\beta_{1j}t_i\right)^{\alpha_j}\right]} \times \prod_{i=N_1+1}^{m}\left(1 + \omega(t_i)\right)^{\alpha_j-1} e^{(R_i+1)\left(1+\omega(t_i)\right)^{\alpha_j}}\end{aligned} \tag{20}$$

where,

$$\nu = \int_0^\infty \int_0^\infty \alpha_j^{n_{ij}-1}\, \beta_{1j}^{n_{1j}+\sigma_{1j}-1} \beta_{2j}^{n_{2j}+\sigma_{2j}-1}\; e^{-(\lambda_{1j}\beta_{1j}+\lambda_{2j}\beta_{2j})} \prod_{i=1}^{N_1} \left(1+\beta_{1j}t_i\right)^{\alpha_j-1}$$
$$\times\, e^{(R_i+1)\left[1-(1+\beta_{1j}t_i)^{\alpha_j}\right]} \prod_{i=N_1+1}^{m} \left(1+\omega(t_i)\right)^{\alpha_j-1} e^{(R_i+1)(1+\omega(t_i))^{\alpha_j}}$$

is normalizing constant.

It may be noted here that the posterior distribution of $(\alpha_j, \beta_{1j}, \beta_{2j})$ takes a ratio form that involves an integration in the denominator and cannot be reduced to a closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of α_j, β_{1j} and β_{2j} will be tedious. Among the various methods suggested to approximate the ratio of integrals of the above form. Therefore, the MCMC technique is used to approximate these integrals.

5.2 Bayesian estimation using the MCMC method

In this subsection, the MCMC method is considered to generate samples from the posterior distribution and then compute the BEs of α_j, β_{1j} and β_{2j} under simple step-stress ALT using progressive Type-II censoring in the presence of competing causes of risks. From the joint posterior density function in Eq. (20), the conditional posterior distributions of α_j, β_{1j} and β_{2j} are given respectively by:

$$\pi^*(\alpha_j|\beta_{1j}\beta_{2j}) \propto \alpha_j^{n_{ij}-1} \prod_{i=1}^{n_1} \left(1+\beta_{1j}t_i\right)^{\alpha_j-1}\; e^{(R_i+1)\left[1-(1+\beta_{1j}t_i)^{\alpha_j}\right]}$$
$$\times \prod_{i=N_1+1}^{m} \left(1+\omega(t_i)\right)^{\alpha_j-1} e^{(R_i+1)(1+\omega(t_i))^{\alpha_j}} \tag{21}$$

$$\pi^*(\beta_{1j}|\alpha_j, \beta_{2j}) \propto \beta_{1j}^{n_{1j}+\sigma_{1j}-1} e^{-\lambda_{1j}\beta_{1j}} \prod_{i=1}^{n_1} \left(1+\beta_{1j}t_i\right)^{\alpha_j-1}\; e^{(R_i+1)\left[1-(1+\beta_{1j}t_i)^{\alpha_j}\right]}$$
$$\times \prod_{i=N_1+1}^{m} \left(1+\omega(t_i)\right)^{\alpha_j-1} e^{(R_i+1)(1+\omega(t_i))^{\alpha_j}} \tag{22}$$

$$\pi^*(\beta_{2j}|\alpha_j, \beta_{1j}) \propto \beta_{2j}^{n_{2j}+\sigma_{2j}-1} e^{-\lambda_{2j}\beta_{2j}} \prod_{i=N_1+1}^{m} \left(1+\omega(t_i)\right)^{\alpha_j-1} e^{(R_i+1)(1+\omega(t_i))^{\alpha_j}} \tag{23}$$

The conditional posterior distributions of α_j, β_{1j} and β_{2j} in Eq. (21), Eq. (22), and Eq. (23) cannot be reduced analytically to well-known distribution. The Metropolis-Hastings algorithm is used to generate random samples from these distributions, see (Upadhyay and Gupta, 2010). For more information concerning the application of M-H, readers may refer to Robert et al. (1999). The following algorithm is proposed to compute Bayes estimators of $U = U(\alpha_j, \beta_{1j}, \beta_{2j})$ under SE and LINEX loss functions.

Algorithm (1)

1. Start with $\alpha_j^{(0)} = \hat{\alpha}_{jMLE}, \beta_{1j}^{(0)} = \hat{\beta}_{1jMLE}, \beta_{2j}^{(0)} = \hat{\beta}_{2jMLE}$
2. Set $i = 1$.
3. Generate $\alpha_j^{(i)}, \beta_{1j}^{(i)}$ and $\beta_{2j}^{(i)}$ from $N(\Theta^{(i-1)}, S_\Theta^2)$, where $\Theta = (\alpha_j, \beta_{1j}, \beta_{2j})$ and S_Θ^2 is found from the Var-Cov matrix for vector Θ.

4. Compute the acceptance ratio $A = \frac{L(\theta^{(i)})\pi(\theta^{(i)})}{L(\theta^{(i-1)})\pi(\theta^{(i-1)})}$
5. Accept the proposed value θ^i with probability $min(A,1)$, Let $\theta^* \to \theta^{(i)}$ (accept), $\theta^* \to \theta^{(i-1)}$ (reject).
6. Set $i = i + 1$.
7. Repeat steps (3)– (6) K times.
8. The approximate means of U and e^{-cU} are given respectively by

$$E(U) = \frac{1}{K-M} \sum_{i=M+1}^{K} U\left(\alpha_j^{(i)}, \beta_{1j}^{(i)}, \beta_{2j}^{(i)}\right), \tag{24}$$

$$E(e^{-cU}) = \frac{1}{K-M} \sum_{i=M+1}^{K} exp\{-cU\left(\alpha_j^{(i)}, \beta_{1j}^{(i)}, \beta_{2j}^{(i)}\right)\}. \tag{25}$$

Where \$ M\$ is the burn-in period. Then, from Eq. (24) and Eq. (25), the Bayes estimators of U under balanced SE and balanced LINEX loss functions are given respectively by

$$\widehat{U}_{SE} = \Omega \widehat{U}_{ML} + (1-\Omega)E(U) \tag{26}$$

$$\widehat{U}_{LINK} = -\frac{1}{C} In\left[\Omega e^{-c\widehat{U}_{ML}} + (1-\Omega)E(e^{-cU})\right] \tag{27}$$

The balanced loss functions are more general, which include the MLE and both symmetric and asymmetric BEs as special cases.

5.3 Highest Posterior Density (HPD) Interval

A $100(1-\gamma)\%$ posterior interval for a random quantity Θ is the interval that has the posterior probability $(1-\gamma)$, that Θ exists in the interval where,

$$P(L \le \Theta \le U) = \int_U^L \pi^*(\Theta|t) d\,\Theta = 1-\gamma$$

The credible CIs of α_j, β_{1j} and β_{2j} are obtained by the following algorithm.

Algorithm (2)

1. Do steps [(1)-(7)] in algorithm (1)
2. Sort the posterior sample $\{\Theta^{(i)}, i = 1,2,\dots,N\}$ to obtain the ordered values as $\Theta^{(1)} \le \Theta^{(2)} \le \cdots . \le \Theta^{(N)}$
3. Determine the number of values to include in the interval $z = [N(1-\gamma)]$
4. Find the smallest interval width by computing $W_i = \Theta^{(i+z)} - \Theta^{(i)}$ Then, $100(1-\gamma)\%$ credible
5. CI of Θ is given by

$$HPD = \{\Theta^{(i_{min})}, \Theta^{(i_{min}+z)}\}$$

6. Simulation Study

In this sub-section, to analyze the performance of estimation methods, including MLE and Bayesian estimation, a Monte Carlo simulation study is employed, under progressive Type-II under step-stress for NH distribution in the presence of competing causes of risks. Following the algorithm (Balakrishnan and Sandhu,1995), 1000 observations are generated from NH distribution based on the following assumptions:

1. Parameters are given by:
   $\beta_{11} = 0.25,\qquad \beta_{12} = 0.20,\qquad \beta_{21} = 0.50, \beta_{22} = 0.45, \alpha_1 = 1.45, \alpha_2 = 1.50$
2. Sample sizes of $n = 60, n = 80$, and $n = 100$.
3. Number of failures of progressive censoring: $m = 30,40,60,80$.

Removed items $R_i$are assumed to be different sample size n and number of failures m as shown in Table (1)

Table 1. Numerous patterns for removing items from life tests at different stages

n	m	Censoring Schemes				
		S_1	S_2	S_3	S_4	S_5
60	30	$(30, 0^{*29})$	$(15, 0^{*28}, 15)$	$(0^{*14},15,15, 0^{*14})$	$(1*^{30}, 0^{*0})$	$(0*^{29}, 30)$
	40	$(20, 0^{*39})$	$(10, 0^{*38}, 10)$	$(0^{*19},10,10, 0^{*19})$	$(1*^{20}, 0^{*20})$	$(0*^{39}, 10)$
80	40	$(40, 0^{*39})$	$(20, 0^{*38}, 20)$	$(0^{*19},20,20, 0^{*19})$	$(1*^{40}, 0^{*0})$	$(0*^{39}, 40)$
	60	$(20, 0^{*59})$	$(10, 0^{*58}, 10)$	$(0^{*29},10,10, 0^{*29})$	$(1*^{20}, 0^{*40})$	$(0*^{59}, 20)$
100	60	$(40, 0^{*59})$	$(20, 0^{*58}, 20)$	$(0^{*29},20,20,0^{*29})$	$(1*^{40}, 0^{*20})$	$(0*^{59}, 40)$
	80	$(20, 0^{*79})$	$(10, 0^{*78}, 10)$	$(0^{*39},10,10, 0^{*39})$	$(1*^{20}, 0^{*60})$	$(0*^{79}, 20)$

Here, $(5^{*3}, 0)$, for example, means that the censoring scheme employed is (5,5,5,0).

Consider the temperature-accelerated levels as $S_1 = 285K, and S_2 = 335K$ and the use temperature is $S_0 = 210K$. The lifetimes of the two failure causes follow NH distributions with shape parameters $\alpha_1 = and \alpha_2 = 1.5$, respectively. If the AFs are given for the failure cause 1 by $ln(\beta_{i1}) = -2.8801 + 425.8185/s_i$ and for failure cause 2 by $ln(\beta_{i2}) = -1.3965 + 200.4975/s_i$ then $\beta_{11} = 0.25, \beta_{12} = 0.20, \beta_{21} = and\ \beta_{22} = 0.4$.

The lifetime of the two failure causes follows an expansion of NH distribution with recognized shape parameter α_1, α_2, respectively. The amount of the parameter was selected to be $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$. To clarify a specific scenario underneath each cause of failure the increase of stress level in our case of the study with the NH model will be achieved by increasing the rate of the scale parameter β_c, which will be reflected in shrinking the main time to failure.

Before continuing, first, the progressively censored Type-II is created through competing risks data utilizing the NH cumulative exposure model (CEM) for constant $m, \left(R_1, R_2, \dots., R_{m,n}\right)$, as shown below:

Step 1: Based on the algorithm proposed by Balakrishnan and Sandhu (1995), generating two samples that are progressively censored $W_1, W_2, \dots W_m$ and $U_1, U_2, \dots U_m$ from Uniform distribution (0,1).

Step 2: Calculating t_{11h} and t_{12h} using

$$U_h = 1 - e^{[1-(1+\beta_{11}t_{11})^{\alpha_1}]}$$

and

$$W_h = 1 - e^{[1-(1+\beta_{12}t_{12})^{\alpha_2}]}$$

the minimum of (t_{11h}, t_{12h}) is registered as t_h^*, while the corresponding minimum index comes out of this condition ($t_{11h} < t_{12h},\ set\ \xi_h^* = 1$; else set $\xi_h^* = 2$) for $1 \le h \le N_1$.

Step 3: Let's assume that various values of stress changing time τ as follows:

$$\tau = \frac{mean(t_h^*) + median(t_h^*)}{2}$$

Step 4: Find N_1 such that $t_{N_1}^* < \tau < t_{N_1+1}^*$. Hence, put $t_h = t_h^*$ and $\xi_h = \xi_h^*$ for $1 \le h \le N_1$.

Step 5: Generating t_{21h} and t_{22h} utilizing

$$U_h = 1 - e^{[1-(1+\beta_{21}(t_{21}-\tau)+\beta_{11}\tau)^{\alpha_1}]}$$

and

$$W_h = 1 - e^{[1-(1+\beta_{22}(t_{22}-\tau)+\beta_{12}\tau)^{\alpha_2}]}$$

for $N_1 + 1 \le h \le m$. The minimum values of (t_{21h}, t_{22h}) assigned as t_h^*.

Step 6: Setting the value of $t_h = t_h^*$ and $\xi_h = \xi_h^*$ for $N_1 + 1 \le h \le m$. Then $t_1, t_2, \dots, t_m$ are the required order observation and $\xi = [\xi_1, \xi_2, \dots, \xi_m]$ the vector of the indices.

MLEs and related 95% asymptotic confidence intervals (Asy-CI) are produced based on the generated data. On deriving MLEs, Be aware that the initial assumed values are regarded as true parameter values.

We compute Bayesian estimates using informative priors for the Bayesian estimation method using 2.5 as a value for all hyperparameters. Such values of informative priors are plugged in to evaluate the required estimates. Through implementing the MH algorithm, the MLEs are used as initial guess values, as well as the corresponding variance-covariance matrix S_θ of $\ln(\hat{\theta})$, where $\theta = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \alpha_1, \alpha_2)$. In the end, 2000 burn-in samples were deleted from the total of 10,000 generated samples by the posterior density and produced Bayes estimates under two loss functions, namely: squared error loss (SEL) function LINEX at $\nu = -0.5, 0.5$ Also, HPD interval estimates have been computed according to the technique of Chen and Shao (1999).

All the average estimates for methods are reported in Table (3) to Table (8) for different combinations of sample size n and number of stages m. Further, the first column donates the average estimates (Avg.) and in the second column, related means square errors (MSEs). For confidence intervals, we have asymptotic confidence intervals for MLEs and HPD for Bayesian estimates based on MCMC which are reported in Table (10) to Table (15) for different combinations of sample size n and a number of stages m. Further, the first column represents the lower bound confidence interval, the second column represents the upper bound of CI, the third column represents the average interval lengths (AILs), and in the last column, related coverage probabilities (CPs) in percentage (%).

From the results Tables [(3)-(8)] we observed the following:

1. The MSE of MLE and BE of the considered parameters decrease as the sample size increases.
2. For the same sample size, the average estimate for MLE tends to the initial values of the parameter with \$m\$ small.
3. For fixed n as m increasing the MLEs for scheme 5 and scheme 3 have the smallest and the largest MSE, respectively.
4. For fixed n as m increasing, the BEs for scheme 1 and scheme 5 have the smallest and the largest MSE, respectively.
5. For fixed n, the MSE of all estimates decreases for all cases as expected with m increasing.
6. The BEs of $\beta_{11}, \beta_{12}, \beta_{21} and \beta_{22}$ under LINEX loss function ($\nu = -0.5$) have the smallest MSEs as compared with estimates under the SE loss function and LINEX loss function ($\nu = 0.5$).

7. The BEs of α_1 and α_2 under LINEX loss function ($\nu = 0.5$) have the smallest MSEs compared to estimates under the SE loss function, LINEX loss function ($\nu = -0.5$).
8. For the complete sample, the MLE and the BEs have the smallest MSE for all cases of progressive censored samples.

Now we compare the average lengths of different confidence/credible intervals. From tables Tables [(10)-(15)], we can see that

1. The length of approximate and credible CIs decreases as the sample size increases.
2. For n increasing, the coverage probability of credible CIs is greater than the coverage probability of approximate.
3. For fixed sample size and m small, the AIL for BE is shorter than the AIL for MLE using MCMC, but in most
   cases, for m increasing and fixed sample size, the AIL for MLE is shorter than the AIL for BEs.

From tables Tables [(17)-(18)], the point and the interval estimates of the parameters α_2, $\beta_{11}, \beta_{21}, \beta_{22}$ for BEs are better than for MLEs and the opposite is true for the parameter α_1, β_{12}.

A comparison of Bayesian estimation under MCMC for all different combinations of sample size n and number of stages m in the case of Scheme 3 (S_3) can be shown graphically The graphs of MCMC estimates for $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \alpha_1$ and α_2 using MH algorithm in cases of informative priors are the plotting of estimates, histogram of estimates, and cumulative mean of estimates. These graphs can be shown in Figure (2) to Figure (4). Also, one can conclude the convergence of estimates under different sample sizes n and the number of stages m.6 .

7. An Illustrative Example

In this section, we simulate progressive Type-II censored samples under step-stress for Nadarajah-Haghighi (NH) distribution in the presence of a competing failure model. The dataset is generated with the following choices of the parameters: $\beta_{11} = 1.75$, $\beta_{12} = 1.80$, $\beta_{21} = 1.65$, $\beta_{22} = 1.70$, $\alpha_1 = 1.20$, and $\alpha_2 = 1.25$ and sample size $n = 50$, number of stages $m = 30$ with censoring scheme:

$$R_i = (2,0,2,0,2,0,2,0,2,0,0,0,0,0,0,0,0,0,0,0,2,0,2,0,2,0,2,0,2).$$

Assumed stress level as $\tau = 0.5$. The progressively censored Type-II data are given in the following Table.

Table 2. The progressively censored Type-II data for an illustrative example

First stress level	(0.004495655,1) (0.016194236,2) (0.024350506,2) (0.025145763,2) (0.040138486,2) (0.042180897,2) (0.053862568,2) (0.056847359,2) (0.066305768,2) (0.080662693,2) (0.166929521,2) (0.176958457,2) (0.178563050,2) (0.184591355,2) (0.192786798,2) (0.208997088,2) (0.218379489,1)
Second stress level	(0.2704664,2) (0.2837838,2) (0.3066917,2) (0.3290515,1) (0.3425723,1) (0.3615052,1) (0.3680463,1) (0.3768950,1) (0.3991376,1) (0.4032060,1) (0.4996059,1) (0.7712304,2) (0.7822973,1)

The first element represents failure time and the second element represents the cause of failure.

From this data set, the estimates (Est.) and standard error (St.Er) of MLEs and BEs using MCMC of the parameters are derived. Also, Asy-CI and HPD intervals are computed. The results are presented in Table (17) for Est. and St.Ers and in Table (18) for Cis (lower, upper, interval length (IL)).

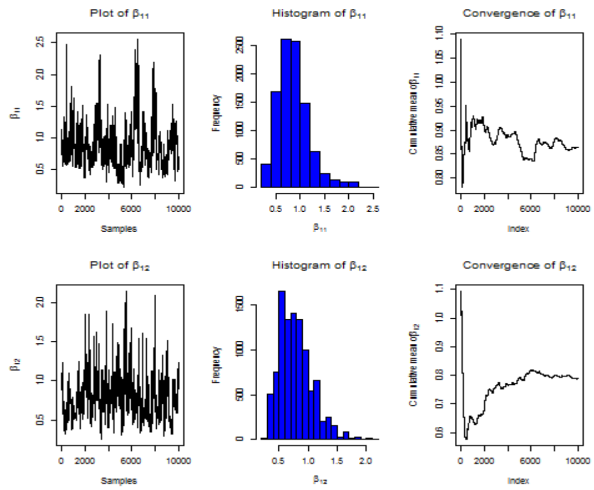

Figure 2. Convergence of MCMC estimates for β_{11} and β_{12} for $n = 60$ and $m = 30$

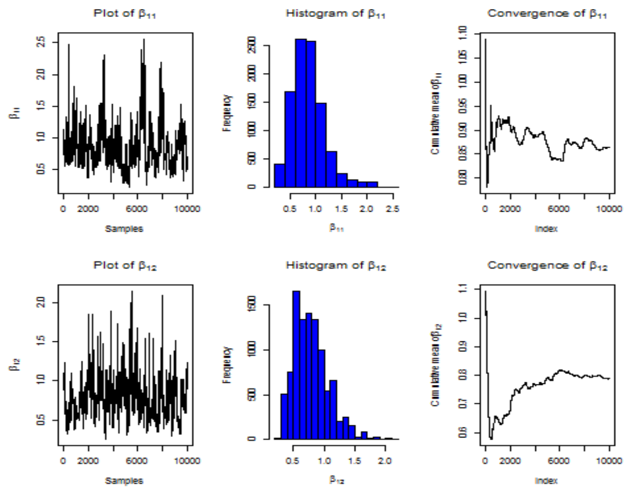

Figure 3. Convergence of MCMC estimates for β_{21} andβ_{22} for $n = 60$ and $m = 30$

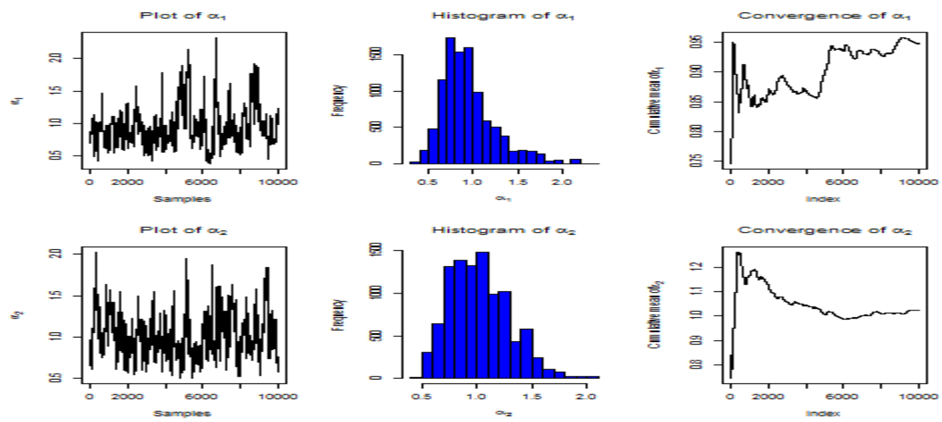

Figure 4. Convergence of MCMC estimates for α_1 and α_2 for $n = 60$ and $m = 30$

Table 3. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes **(Sch.)** and parameters **(Parm.)** of progressive Type-II censoring step-stress for NH distribution at $n = 60$ and $m = 30$

Sch.	**Parm.**	**MLE**		**BE MCMC: SEL**		**BE MCMC: LINEX**			
						$\boldsymbol{v = -0.5}$		$\boldsymbol{v = 0.5}$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
$\boldsymbol{S_1}$	$\boldsymbol{\alpha_1}$	1.5257	0.0750	1.0918	0.1485	1.0604	0.1681	1.1296	0.1297
	$\boldsymbol{\alpha_2}$	1.5290	0.0792	1.0822	0.1938	1.0522	0.2162	1.1183	0.1709
	$\boldsymbol{\beta_{11}}$	0.2886	0.0068	0.5098	0.0798	0.4974	0.0724	0.5233	0.0884
	$\boldsymbol{\beta_{12}}$	0.2903	0.0150	0.5126	0.1099	0.5005	0.1014	0.5259	0.1196
	$\boldsymbol{\beta_{21}}$	0.6295	0.1064	1.2730	0.6596	1.1639	0.4883	1.4259	0.9540
	$\boldsymbol{\beta_{22}}$	0.6437	0.1656	1.2875	0.7623	1.1782	0.5774	1.4392	1.0693
$\boldsymbol{S_2}$	$\boldsymbol{\alpha_1}$	1.5010	0.0643	0.9817	0.2396	0.9476	0.2687	1.0235	0.2095
	$\boldsymbol{\alpha_2}$	1.5377	0.1011	0.9822	0.2918	0.9478	0.3239	1.0248	0.2577
	$\boldsymbol{\beta_{11}}$	0.2997	0.0113	0.5879	0.1319	0.5696	0.1180	0.6086	0.1487
	$\boldsymbol{\beta_{12}}$	0.2932	0.0153	0.5887	0.1694	0.5701	0.1533	0.6098	0.1889
	$\boldsymbol{\beta_{21}}$	0.5938	0.1203	1.2674	0.6496	1.1619	0.4857	1.4137	0.9228
	$\boldsymbol{\beta_{22}}$	0.5819	0.1071	1.2733	0.7432	1.1665	0.5636	1.4220	1.0428
$\boldsymbol{S_3}$	$\boldsymbol{\alpha_1}$	1.5114	0.0922	1.0865	0.1536	1.0519	0.1753	1.1283	0.1336

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v = -0.5$		$v = 0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
	α_2	1.5280	0.1027	1.0881	0.1914	1.0538	0.2166	1.1295	0.1661
	β_{11}	0.2966	0.0095	0.4948	0.0704	0.4837	0.0643	0.5068	0.0774
	β_{12}	0.2945	0.0169	0.4927	0.0963	0.4819	0.0893	0.5045	0.1044
	β_{21}	0.6783	0.1822	1.2708	0.6503	1.1630	0.4834	1.4201	0.9300
	β_{22}	0.6775	0.2122	1.2675	0.7368	1.1612	0.5587	1.4185	1.0573
S_4	α_1	1.5012	0.0698	1.0376	0.1911	1.0015	0.2182	1.0816	0.1633
	α_2	1.5168	0.0948	1.0381	0.2322	1.0019	0.2633	1.0824	0.1997
	β_{11}	0.2942	0.0080	0.5314	0.0923	0.5171	0.0832	0.5473	0.1031
	β_{12}	0.2934	0.0153	0.5299	0.1209	0.5155	0.1104	0.5457	0.1331
	β_{21}	0.6511	0.1142	1.2819	0.6742	1.1752	0.5045	1.4282	0.9534
	β_{22}	0.6560	0.1498	1.2790	0.7468	1.1712	0.5658	1.4287	1.0473
S_5	α_1	1.5156	0.0811	0.9652	0.2630	0.9276	0.2954	1.0112	0.2293
	α_2	1.5539	0.0976	0.9671	0.3100	0.9288	0.3468	1.0147	0.2715
	β_{11}	0.2950	0.0078	0.5961	0.1366	0.5762	0.1212	0.6188	0.1553
	β_{12}	0.2868	0.0125	0.5972	0.1759	0.5773	0.1585	0.6198	0.1968
	β_{21}	0.5825	0.0652	1.2602	0.6529	1.1571	0.4890	1.4033	0.9278
	β_{22}	0.5668	0.0656	1.2527	0.7063	1.1496	0.5370	1.3961	0.9879

Table 4. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for NH distribution at $n = 60$ and $m = 40$

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v = -0.5$		$v = 0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	α_1	1.4951	0.0662	1.0966	0.1457	1.0657	0.1641	1.1343	0.1286
	α_2	1.5181	0.1246	1.1043	0.1797	1.0747	0.1996	1.1390	0.1603
	β_{11}	0.2901	0.0074	0.4903	0.0671	0.4797	0.0613	0.5017	0.0736
	β_{12}	0.2887	0.0143	0.4846	0.0905	0.4744	0.0841	0.4955	0.0978
	β_{21}	0.6482	0.1409	1.2668	0.6574	1.1611	0.4898	1.4116	0.9348
	β_{22}	0.6495	0.1693	1.2450	0.7010	1.1437	0.5343	1.3826	0.9710
S_2	α_1	1.4784	0.0491	1.0239	0.2032	0.9909	0.2277	1.0644	0.1791
	α_2	1.4961	0.0761	1.0177	0.2556	0.9852	0.2830	1.0574	0.2277
	β_{11}	0.2906	0.0059	0.5347	0.0923	0.5211	0.0837	0.5496	0.1023
	β_{12}	0.2908	0.0145	0.5375	0.1256	0.5240	0.1157	0.5521	0.1370
	β_{21}	0.5944	0.0562	1.2618	0.6466	1.1583	0.4852	1.4035	0.9132
	β_{22}	0.6042	0.1213	1.2730	0.7449	1.1702	0.5719	1.4113	1.0201
S_3	α_1	1.4866	0.0804	1.0805	0.1565	1.0492	0.1773	1.1183	0.1356
	α_2	1.4927	0.0965	1.0876	0.1935	1.0556	0.2167	1.1264	0.1706
	β_{11}	0.2902	0.0076	0.4774	0.0590	0.4678	0.0542	0.4877	0.0644
	β_{12}	0.2915	0.0153	0.4751	0.0840	0.4657	0.0783	0.4853	0.0904
	β_{21}	0.6701	0.1469	1.3163	1.3044	1.1718	0.5008	1.4246	0.9578
	β_{22}	0.6815	0.1968	1.2649	0.7252	1.1714	0.6091	1.4095	1.0125
S_4	α_1	1.5029	0.0674	1.1046	0.1417	1.0722	0.1607	1.1434	0.1237
	α_2	1.5139	0.0758	1.1065	0.1747	1.0752	0.1965	1.1437	0.1529
	β_{11}	0.2874	0.0064	0.4787	0.0619	0.4690	0.0568	0.4892	0.0676
	β_{12}	0.2875	0.0140	0.4741	0.0845	0.4648	0.0788	0.4840	0.0908
	β_{21}	0.6361	0.1150	1.2416	0.6145	1.1400	0.4605	1.3812	0.8691
	β_{22}	0.6421	0.1471	1.2252	0.6630	1.1278	0.5082	1.3573	0.9124

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX $v=-0.5$		BE MCMC: LINEX $v=0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_5	α_1	1.5019	0.0484	0.9969	0.2279	0.9641	0.2545	1.0361	0.2005
	α_2	1.5275	0.0779	1.0121	0.2656	0.9748	0.2964	1.0595	0.2363
	β_{11}	0.2853	0.0046	0.5506	0.1035	0.5359	0.0937	0.5668	0.1150
	β_{12}	0.2848	0.0125	0.5458	0.1323	0.5308	0.1209	0.5622	0.1455
	β_{21}	0.5778	0.0375	1.2636	0.6475	1.1639	0.4914	1.3993	0.9010
	β_{22}	0.5838	0.0712	1.2528	0.7076	1.1509	0.5397	1.3921	0.9824

Table 5. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for NH distribution at $n = 80$ and $m = 40$

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX $v=-0.5$		BE MCMC: LINEX $v=0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	α_1	1.5212	0.0801	1.0984	0.1441	1.0676	0.1622	1.1361	0.1280
	α_2	1.5396	0.0950	1.1033	0.1799	1.0728	0.2005	1.1395	0.1602
	β_{11}	0.2846	0.0064	0.4901	0.0673	0.4797	0.0617	0.5014	0.0738
	β_{12}	0.2833	0.0132	0.4894	0.0957	0.4788	0.0887	0.5009	0.1038
	β_{21}	0.6134	0.0800	1.2479	0.6248	1.1461	0.4690	1.3876	0.8829
	β_{22}	0.6156	0.1145	1.2429	0.7011	1.1409	0.5334	1.3821	0.9731
S_2	α_1	1.5206	0.0704	1.0047	0.2223	0.9686	0.2506	1.0501	0.1945
	α_2	1.5271	0.0708	1.0049	0.2694	0.9694	0.3007	1.0490	0.2370
	β_{11}	0.2896	0.0150	0.5568	0.1090	0.5415	0.0983	0.5738	0.1215
	β_{12}	0.2884	0.0138	0.5561	0.1406	0.5410	0.1288	0.5727	0.1542
	β_{21}	0.5869	0.2473	1.2649	0.6506	1.1623	0.4901	1.4056	0.9136
	β_{22}	0.5887	0.1275	1.2596	0.7161	1.1577	0.5470	1.3998	0.9966
S_3	α_1	1.5176	0.0970	1.0945	0.1475	1.0613	0.1681	1.1345	0.1275
	α_2	1.5225	0.0949	1.0912	0.1890	1.0593	0.2118	1.1295	0.1666
	β_{11}	0.2898	0.0090	0.4716	0.0568	0.4628	0.0525	0.4810	0.0616
	β_{12}	0.2894	0.0152	0.4711	0.0807	0.4624	0.0757	0.4804	0.0864
	β_{21}	0.6672	0.2007	1.2526	0.6345	1.1507	0.4748	1.4011	1.0493
	β_{22}	0.6695	0.2081	1.2494	0.7019	1.1507	0.5393	1.3843	0.9659
S_4	α_1	1.4865	0.0715	1.0403	0.1902	1.0051	0.2154	1.0833	0.1653
	α_2	1.5383	0.1329	1.0305	0.2429	0.9967	0.2714	1.0712	0.2136
	β_{11}	0.2907	0.0078	0.5085	0.0766	0.4967	0.0699	0.5214	0.0845
	β_{12}	0.2848	0.0143	0.5139	0.1083	0.5019	0.1001	0.5269	0.1176
	β_{21}	0.6414	0.0982	1.2595	0.6367	1.1590	0.4807	1.3972	0.8932
	β_{22}	0.6354	0.1448	1.2795	0.7584	1.1767	0.5829	1.4189	1.0383
S_5	α_1	1.5104	0.0565	0.9605	0.2629	0.9241	0.2949	1.0054	0.2298
	α_2	1.5576	0.0744	0.9602	0.3180	0.9233	0.3533	1.0062	0.2813
	β_{11}	0.2888	0.0052	0.5817	0.1237	0.5644	0.1112	0.6009	0.1386
	β_{12}	0.2804	0.0099	0.5865	0.1654	0.5687	0.1504	0.6063	0.1831
	β_{21}	0.5638	0.0395	1.2480	0.6218	1.1505	0.4718	1.3790	0.8611
	β_{22}	0.5466	0.0423	1.2540	0.7114	1.1542	0.5459	1.3908	0.9814

Table 6. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for NH distribution at $n = 80$ and $m = 60$

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v = -0.5$		$v = 0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	α_1	1.5166	0.0702	1.1209	0.1292	1.0921	0.1448	1.1555	0.1153
	α_2	1.5308	0.0931	1.1167	0.1706	1.0891	0.1886	1.1490	0.1529
	β_{11}	0.2784	0.0050	0.4559	0.0487	0.4483	0.0452	0.4639	0.0524
	β_{12}	0.2789	0.0113	0.4599	0.0750	0.4521	0.0706	0.4682	0.0799
	β_{21}	0.6077	0.0794	1.1942	0.5462	1.1065	0.4186	1.3098	0.7460
	β_{22}	0.6186	0.1246	1.2064	0.6458	1.1161	0.5012	1.3279	0.8775
S_2	α_1	1.4882	0.0624	1.0703	0.1697	1.0389	0.1896	1.1078	0.1512
	α_2	1.5018	0.0770	1.0630	0.2158	1.0324	0.2388	1.0995	0.1929
	β_{11}	0.2917	0.0096	0.4952	0.0679	0.4851	0.0625	0.5061	0.0740
	β_{12}	0.2902	0.0149	0.4990	0.0990	0.4892	0.0926	0.5094	0.1060
	β_{21}	0.6127	0.2074	1.2015	0.5619	1.1110	0.4287	1.3239	0.7789
	β_{22}	0.6100	0.1596	1.2058	0.6403	1.1168	0.4979	1.3235	0.8628
S_3	α_1	1.5008	0.0619	1.1054	0.1373	1.0758	0.1550	1.1404	0.1203
	α_2	1.5142	0.0817	1.1125	0.1736	1.0826	0.1934	1.1481	0.1546
	β_{11}	0.2836	0.0058	0.4584	0.0494	0.4509	0.0459	0.4664	0.0532
	β_{12}	0.2838	0.0126	0.4575	0.0736	0.4499	0.0694	0.4655	0.0783
	β_{21}	0.6168	0.0811	1.2027	0.5551	1.1133	0.4244	1.3211	0.7617
	β_{22}	0.6269	0.1306	1.1918	0.6179	1.1031	0.4779	1.3210	0.9516
S_4	α_1	1.5121	0.0776	1.1225	0.1300	1.0942	0.1451	1.1555	0.1160
	α_2	1.5315	0.0857	1.1183	0.1685	1.0888	0.1872	1.1536	0.1509
	β_{11}	0.2809	0.0056	0.4511	0.0461	0.4439	0.0429	0.4587	0.0496
	β_{12}	0.2794	0.0119	0.4561	0.0722	0.4486	0.0681	0.4640	0.0766
	β_{21}	0.6156	0.0999	1.1804	0.5272	1.0951	0.4052	1.2926	0.7170
	β_{22}	0.6182	0.1448	1.1942	0.6169	1.1057	0.4798	1.3114	0.8307
S_5	α_1	1.4948	0.0490	1.0334	0.1972	1.0011	0.2203	1.0727	0.1745
	α_2	1.5020	0.0587	1.0245	0.2512	0.9919	0.2776	1.0645	0.2254
	β_{11}	0.2866	0.0050	0.5189	0.0821	0.5074	0.0753	0.5313	0.0898
	β_{12}	0.2879	0.0138	0.5253	0.1165	0.5136	0.1083	0.5378	0.1258
	β_{21}	0.5685	0.0397	1.2109	0.5766	1.1198	0.4394	1.3331	0.7975
	β_{22}	0.5745	0.0683	1.2272	0.6623	1.1348	0.5158	1.3481	0.8861

Table 7. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for NH distribution at $n = 100$ and $m = 60$

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v = -0.5$		$v = 0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	α_1	1.5105	0.0830	1.1051	0.1418	1.0787	0.1567	1.1354	0.1277
	α_2	1.5212	0.0851	1.1255	0.1654	1.0961	0.1835	1.1602	0.1484
	β_{11}	0.2837	0.0065	0.4657	0.0530	0.4578	0.0492	0.4742	0.0572
	β_{12}	0.2826	0.0125	0.4594	0.0741	0.4514	0.0697	0.4678	0.0790
	β_{21}	0.6327	0.1484	1.2219	0.5962	1.1308	0.4565	1.3465	0.8287
	β_{22}	0.6281	0.1423	1.1933	0.6221	1.1034	0.4809	1.3162	0.8548

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v=-0.5$		$v=0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_2	α_1	1.4943	0.0571	1.0294	0.2016	0.9975	0.2246	1.0685	0.1792
	α_2	1.5100	0.0737	1.0228	0.2531	0.9914	0.2790	1.0604	0.2269
	β_{11}	0.2863	0.0052	0.5183	0.0821	0.5070	0.0754	0.5304	0.0898
	β_{12}	0.2862	0.0127	0.5217	0.1134	0.5105	0.1057	0.5337	0.1221
	β_{21}	0.5750	0.0523	1.2113	0.5755	1.1269	0.4575	1.3292	0.7846
	β_{22}	0.5784	0.0789	1.2279	0.6893	1.1360	0.5308	1.3554	1.0191
S_3	α_1	1.5082	0.0870	1.1151	0.1342	1.0847	0.1515	1.1510	0.1178
	α_2	1.5144	0.1042	1.1184	0.1687	1.0871	0.1893	1.1559	0.1485
	β_{11}	0.2810	0.0057	0.4436	0.0434	0.4369	0.0405	0.4506	0.0465
	β_{12}	0.2819	0.0120	0.4432	0.0647	0.4364	0.0611	0.4505	0.0686
	β_{21}	0.6316	0.0971	1.1847	0.5314	1.0999	0.4095	1.2952	0.7194
	β_{22}	0.6396	0.1357	1.1857	0.6065	1.0984	0.4718	1.3067	0.8312
S_4	α_1	1.5105	0.0798	1.1148	0.1352	1.0854	0.1513	1.1498	0.1207
	α_2	1.5143	0.0936	1.1162	0.1671	1.0858	0.1879	1.1523	0.1466
	β_{11}	0.2847	0.0063	0.4530	0.0473	0.4460	0.0442	0.4605	0.0508
	β_{12}	0.2860	0.0131	0.4547	0.0712	0.4472	0.0670	0.4626	0.0759
	β_{21}	0.6243	0.1060	1.1773	0.5265	1.0944	0.4068	1.2873	0.7171
	β_{22}	0.6349	0.1433	1.1860	0.6133	1.0978	0.4743	1.3046	0.8375
S_5	α_1	1.4968	0.0441	0.9991	0.2316	0.9642	0.2585	1.0414	0.2047
	α_2	1.5467	0.0896	1.0069	0.2712	0.9709	0.3021	1.0516	0.2402
	β_{11}	0.2837	0.0044	0.5395	0.0946	0.5264	0.0863	0.5536	0.1039
	β_{12}	0.2774	0.0100	0.5359	0.1249	0.5227	0.1152	0.5503	0.1360
	β_{21}	0.5702	0.0347	1.2472	0.6362	1.1523	0.4867	1.3733	0.8716
	β_{22}	0.5626	0.0565	1.2323	0.6824	1.1391	0.5306	1.3571	0.9226

Table 8. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for NH distribution at $n = 100$ and $m = 80$

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v=-0.5$		$v=0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	α_1	1.4890	0.0634	1.1435	0.1176	1.1160	0.1310	1.1756	0.1056
	α_2	1.5003	0.0739	1.1439	0.1520	1.1159	0.1681	1.1767	0.1371
	β_{11}	0.2856	0.0070	0.4385	0.0416	0.4322	0.0390	0.4452	0.0445
	β_{12}	0.2841	0.0126	0.4390	0.0627	0.4328	0.0595	0.4456	0.0661
	β_{21}	0.6291	0.1252	1.1383	0.4796	1.0609	0.3717	1.2411	0.6506
	β_{22}	0.6287	0.1470	1.1392	0.5485	1.0633	0.4352	1.2367	0.7184
S_2	α_1	1.4935	0.0587	1.0809	0.1633	1.0508	0.1814	1.1170	0.1467
	α_2	1.5106	0.0699	1.0906	0.1942	1.0591	0.2153	1.1283	0.1748
	β_{11}	0.2824	0.0051	0.4772	0.0597	0.4688	0.0555	0.4861	0.0643
	β_{12}	0.2811	0.0118	0.4726	0.0814	0.4643	0.0765	0.4815	0.0868
	β_{21}	0.5864	0.0716	1.1857	0.5529	1.0991	0.4235	1.3006	0.7594
	β_{22}	0.5893	0.1075	1.1674	0.5920	1.0825	0.4597	1.2794	0.8007
S_3	α_1	1.4863	0.0780	1.1358	0.1266	1.1087	0.1401	1.1669	0.1143
	α_2	1.4898	0.0841	1.1320	0.1606	1.1033	0.1781	1.1661	0.1440

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v=-0.5$		$v=0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
	β_{11}	0.2867	0.0067	0.4363	0.0402	0.4303	0.0378	0.4425	0.0428
	β_{12}	0.2882	0.0141	0.4393	0.0627	0.4330	0.0595	0.4458	0.0661
	β_{21}	0.6431	0.1470	1.1427	0.4903	1.0676	0.3839	1.2398	0.6527
	β_{22}	0.6552	0.1932	1.1552	0.5701	1.0772	0.4520	1.2551	0.7448
S_4	α_1	1.5167	0.0752	1.1359	0.1214	1.1088	0.1355	1.1671	0.1082
	α_2	1.5178	0.0777	1.1393	0.1551	1.1110	0.1716	1.1728	0.1398
	β_{11}	0.2747	0.0042	0.4353	0.0391	0.4292	0.0366	0.4418	0.0417
	β_{12}	0.2771	0.0108	0.4349	0.0606	0.4288	0.0576	0.4412	0.0639
	β_{21}	0.6012	0.0821	1.1541	0.5043	1.0751	0.3915	1.2581	0.6816
	β_{22}	0.6163	0.1345	1.1496	0.5619	1.0717	0.4442	1.2500	0.7389
S_5	α_1	1.4921	0.0503	1.0727	0.1743	1.0392	0.1938	1.1135	0.1578
	α_2	1.5119	0.0637	1.0656	0.2119	1.0334	0.2363	1.1042	0.1876
	β_{11}	0.2831	0.0047	0.4874	0.0654	0.4783	0.0606	0.4972	0.0707
	β_{12}	0.2814	0.0112	0.4885	0.0906	0.4792	0.0849	0.4985	0.0970
	β_{21}	0.5775	0.0493	1.1811	0.5374	1.0970	0.4149	1.2914	0.7279
	β_{22}	0.5814	0.0882	1.1846	0.6055	1.0981	0.4724	1.2990	0.8139

Table 9. Avg. estimated values and MSEs of the ML and BE using MCMC at complete sampling step-stress for NH distribution at $n=60, 80$ and $n=100$

n	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						$v=-0.5$		$v=0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
60	α_1	1.5123	0.0708	1.1304	0.1265	1.1008	0.1412	1.1657	0.1152
	α_2	1.5252	0.0935	1.1137	0.1716	1.0871	0.1889	1.1446	0.1547
	β_{11}	0.2807	0.0060	0.4524	0.0471	0.4450	0.0438	0.4603	0.0507
	β_{12}	0.2808	0.0124	0.4585	0.0732	0.4510	0.0691	0.4665	0.0778
	β_{21}	0.6168	0.1007	1.1778	0.5196	1.0915	0.3974	1.2923	0.7142
	β_{22}	0.6260	0.1648	1.2001	0.6320	1.1133	0.4948	1.3165	0.8492
80	α_1	1.4996	0.0658	1.1379	0.1223	1.1106	0.1357	1.1696	0.1103
	α_2	1.5076	0.0850	1.1438	0.1520	1.1156	0.1682	1.1771	0.1377
	β_{11}	0.2815	0.0051	0.4431	0.0437	0.4366	0.0409	0.4500	0.0468
	β_{12}	0.2832	0.0122	0.4398	0.0636	0.4334	0.0602	0.4466	0.0672
	β_{21}	0.6012	0.0745	1.1310	0.4623	1.0547	0.3595	1.2311	0.6227
	β_{22}	0.6140	0.1165	1.1244	0.5231	1.0491	0.4145	1.2229	0.6900
100	α_1	1.5110	0.0674	1.1539	0.1173	1.1277	0.1288	1.1842	0.1074
	α_2	1.5282	0.0791	1.1585	0.1448	1.1317	0.1588	1.1898	0.1329
	β_{11}	0.2768	0.0049	0.4272	0.0369	0.4218	0.0348	0.4329	0.0391
	β_{12}	0.2750	0.0103	0.4244	0.0555	0.4191	0.0529	0.4300	0.0582
	β_{21}	0.5973	0.0911	1.1019	0.4446	1.0357	0.3545	1.1855	0.5780
	β_{22}	0.5963	0.1171	1.0916	0.4872	1.0251	0.3907	1.1752	0.6284

Table 10. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for NH distribution at $n = 60$ and $m = 30$

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	α_1	1.0096	2.0418	1.0322	96.1	0.8225	1.3720	0.5495	96.0
	α_2	0.9800	2.0779	1.0980	95.9	0.8481	1.3675	0.5195	97.5
	β_{11}	0.1452	0.4319	0.2866	96.4	0.3055	0.7195	0.4140	95.6
	β_{12}	0.1280	0.4526	0.3246	96.0	0.3271	0.7311	0.4040	96.8
	β_{21}	0.0425	1.2165	1.1740	96.2	0.8014	1.7229	0.9215	96.4
	β_{22}	0.0000	1.3456	1.3456	96.5	0.8067	1.7385	0.9318	97.1
S_2	α_1	1.0139	1.9881	0.9742	97.0	0.7250	1.2625	0.5375	96.1
	α_2	0.9188	2.1567	1.2379	97.4	0.7181	1.2803	0.5622	96.1
	β_{11}	0.1153	0.4842	0.3689	97.4	0.3480	0.8266	0.4786	96.0
	β_{12}	0.1334	0.4529	0.3195	97.3	0.3462	0.8506	0.5044	96.3
	β_{21}	0.0000	1.2487	1.2487	98.3	0.7952	1.7345	0.9393	96.8
	β_{22}	0.0000	1.1693	1.1693	96.4	0.7705	1.7428	0.9722	97.4
S_3	α_1	0.9284	2.0944	1.1660	96.0	0.8355	1.3675	0.5320	96.3
	α_2	0.9020	2.1540	1.2520	96.5	0.8183	1.3663	0.5480	96.5
	β_{11}	0.1287	0.4644	0.3357	96.5	0.3071	0.6893	0.3822	96.3
	β_{12}	0.1198	0.4693	0.3495	96.1	0.3217	0.7029	0.3811	96.4
	β_{21}	0.0000	1.4389	1.4389	97.1	0.8231	1.7147	0.8916	96.4
	β_{22}	0.0000	1.4629	1.4629	96.2	0.7900	1.7764	0.9864	96.8
S_4	α_1	0.9928	2.0096	1.0168	97.3	0.7837	1.3280	0.5443	96.3
	α_2	0.9140	2.1195	1.2055	97.1	0.7992	1.3299	0.5307	97.4
	β_{11}	0.1421	0.4463	0.3042	95.2	0.3425	0.7548	0.4123	96.3
	β_{12}	0.1350	0.4518	0.3168	96.2	0.3296	0.7391	0.4095	95.7
	β_{21}	0.0585	1.2438	1.1853	96.7	0.8029	1.7595	0.9567	96.2
	β_{22}	0.0134	1.2986	1.2852	96.4	0.8590	1.7497	0.8907	97.6
S_5	α_1	0.9722	2.0589	1.0867	96.2	0.6865	1.3103	0.6237	96.2
	α_2	0.9503	2.1575	1.2072	96.6	0.6966	1.3009	0.6043	96.1
	β_{11}	0.1464	0.4436	0.2972	96.6	0.3744	0.8431	0.4688	97.0
	β_{12}	0.1489	0.4248	0.2759	97.0	0.3484	0.8312	0.4829	95.7
	β_{21}	0.1087	1.0563	0.9476	97.7	0.7624	1.8282	1.0658	97.0
	β_{22}	0.1201	1.0136	0.8935	96.6	0.7807	1.7331	0.9523	96.5

Table 11. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for NH distribution at $n = 60$ and $m = 40$

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	α_1	0.9982	1.9919	0.9938	97.4	0.8491	1.3825	0.5334	96.1
	α_2	0.8268	2.2094	1.3826	97.8	0.8476	1.4023	0.5547	95.8
	β_{11}	0.1408	0.4394	0.2986	95.3	0.3262	0.6810	0.3548	97.1
	β_{12}	0.1314	0.4459	0.3144	95.1	0.3135	0.6820	0.3685	97.3
	β_{21}	0.0000	1.3244	1.3244	95.5	0.8381	1.8435	1.0055	98.5
	β_{22}	0.0000	1.3551	1.3551	95.6	0.7905	1.8166	1.0261	98.1
S_2	α_1	1.0476	1.9093	0.8617	97.4	0.7796	1.3316	0.5521	96.9
	α_2	0.9551	2.0372	1.0821	98.5	0.7344	1.2955	0.5611	95.3

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
	β_{11}	0.1632	0.4180	0.2549	96.1	0.3365	0.7229	0.3864	96.3
	β_{12}	0.1357	0.4459	0.3102	96.5	0.3409	0.7433	0.4024	96.2
	β_{21}	0.1682	1.0207	0.8525	97.0	0.7804	1.7697	0.9893	97.5
	β_{22}	0.0000	1.2165	1.2165	97.7	0.7510	1.7810	1.0300	96.9
S_3	α_1	0.9352	2.0379	1.1027	96.6	0.8255	1.3639	0.5384	96.3
	α_2	0.8837	2.1016	1.2180	96.8	0.8492	1.4023	0.5531	97.0
	β_{11}	0.1387	0.4416	0.3029	95.0	0.3112	0.6428	0.3315	96.4
	β_{12}	0.1280	0.4551	0.3271	94.5	0.3085	0.6500	0.3415	96.4
	β_{21}	0.0000	1.3435	1.3435	95.8	0.7689	1.6926	0.9237	95.5
	β_{22}	0.0000	1.4236	1.4236	96.1	0.8322	1.7614	0.9292	97.6
S_4	α_1	1.0043	2.0015	0.9972	96.1	0.8255	1.3971	0.5716	95.6
	α_2	0.9746	2.0532	1.0786	96.5	0.8642	1.3730	0.5089	96.7
	β_{11}	0.1487	0.4261	0.2774	96.1	0.3196	0.7023	0.3826	97.4
	β_{12}	0.1309	0.4441	0.3132	95.8	0.3093	0.6747	0.3653	96.3
	β_{21}	0.0268	1.2453	1.2185	96.0	0.8153	1.7675	0.9522	98.1
	β_{22}	0.0000	1.2931	1.2931	95.6	0.7897	1.7073	0.9177	97.2
S_5	α_1	1.0825	1.9213	0.8388	96.1	0.7312	1.2769	0.5457	96.3
	α_2	0.9829	2.0720	1.0891	95.5	0.7157	1.3185	0.6028	96.1
	β_{11}	0.1711	0.3994	0.2282	96.0	0.3447	0.7727	0.4280	96.3
	β_{12}	0.1415	0.4281	0.2865	96.2	0.3432	0.7643	0.4211	96.8
	β_{21}	0.2301	0.9255	0.6954	95.7	0.8051	1.7721	0.9670	97.2
	β_{22}	0.1312	1.0363	0.9051	95.5	0.7931	1.7514	0.9583	96.4

Table 12. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for NH distribution at $n = 80$ and $m = 40$

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	α_1	0.9841	2.0582	1.0742	94.6	0.8573	1.4023	0.5450	97.0
	α_2	0.9401	2.1391	1.1990	96.1	0.8185	1.3714	0.5529	95.7
	β_{11}	0.1437	0.4254	0.2817	95.6	0.3254	0.6856	0.3602	97.0
	β_{12}	0.1276	0.4390	0.3114	95.8	0.2981	0.7245	0.4264	96.1
	β_{21}	0.1053	1.1216	1.0163	95.8	0.7313	1.7207	0.9894	96.4
	β_{22}	0.0368	1.1943	1.1575	94.8	0.7552	1.7803	1.0251	97.5
S_2	α_1	1.0191	2.0221	1.0030	96.6	0.7403	1.3208	0.5805	96.0
	α_2	1.0082	2.0460	1.0378	96.9	0.7128	1.2995	0.5867	95.8
	β_{11}	0.0627	0.5165	0.4538	99.5	0.3463	0.8160	0.4697	97.4
	β_{12}	0.1366	0.4403	0.3037	97.3	0.3374	0.7956	0.4582	96.3
	β_{21}	0.0000	1.5470	1.5470	99.7	0.7730	1.7377	0.9646	96.5
	β_{22}	0.0000	1.2338	1.2338	97.2	0.7849	1.7167	0.9317	96.5
S_3	α_1	0.9216	2.1136	1.1921	96.3	0.8468	1.3863	0.5395	96.5
	α_2	0.9200	2.1250	1.2050	96.6	0.8449	1.3896	0.5447	96.0
	β_{11}	0.1213	0.4582	0.3369	96.4	0.3124	0.6487	0.3363	96.4
	β_{12}	0.1224	0.4564	0.3340	95.4	0.3182	0.6378	0.3197	96.9
	β_{21}	0.0000	1.4823	1.4823	96.6	0.7532	1.6851	0.9319	96.2
	β_{22}	0.0000	1.4537	1.4537	95.3	0.7562	1.7212	0.9650	96.5

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_4	α_1	0.9669	2.0061	1.0392	96.6	0.7636	1.3149	0.5513	95.4
	α_2	0.8274	2.2493	1.4218	97.4	0.7824	1.3468	0.5644	97.4
	β_{11}	0.1371	0.4444	0.3073	94.5	0.3419	0.7098	0.3680	96.7
	β_{12}	0.1189	0.4506	0.3317	95.9	0.3422	0.7033	0.3611	96.7
	β_{21}	0.0930	1.1897	1.0967	96.0	0.8087	1.7184	0.9097	96.9
	β_{22}	0.0000	1.2871	1.2871	96.0	0.7820	1.7523	0.9703	96.4
S_5	α_1	1.0594	1.9613	0.9019	95.7	0.6827	1.2666	0.5839	96.6
	α_2	1.0349	2.0803	1.0453	95.7	0.6985	1.3158	0.6173	97.2
	β_{11}	0.1699	0.4078	0.2379	96.6	0.3686	0.8304	0.4619	96.9
	β_{12}	0.1647	0.3960	0.2314	97.2	0.3584	0.8355	0.4770	96.4
	β_{21}	0.1945	0.9331	0.7386	97.0	0.7695	1.7339	0.9644	96.4
	β_{22}	0.1905	0.9027	0.7122	95.2	0.7726	1.7411	0.9685	96.4

Table 13. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for NH distribution at $n = 80$ and $m = 60$

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	α_1	1.0139	2.0193	1.0055	95.9	0.8704	1.4061	0.5357	95.8
	α_2	0.9355	2.1261	1.1906	95.7	0.8587	1.4181	0.5593	96.0
	β_{11}	0.1509	0.4059	0.2549	95.1	0.3126	0.6013	0.2887	96.4
	β_{12}	0.1397	0.4180	0.2783	94.3	0.3025	0.6161	0.3136	96.2
	β_{21}	0.0972	1.1182	1.0210	95.4	0.7345	1.6924	0.9579	96.9
	β_{22}	0.0104	1.2268	1.2163	95.5	0.7076	1.7118	1.0042	97.2
S_2	α_1	1.0042	1.9722	0.9680	96.9	0.7759	1.3670	0.5911	95.4
	α_2	0.9576	2.0460	1.0884	96.6	0.7749	1.3501	0.5752	95.5
	β_{11}	0.1178	0.4656	0.3478	97.7	0.3352	0.6686	0.3334	96.7
	β_{12}	0.1292	0.4511	0.3219	96.7	0.3271	0.6951	0.3680	96.7
	β_{21}	0.0000	1.4779	1.4779	97.9	0.7185	1.7328	1.0143	98.1
	β_{22}	0.0000	1.3277	1.3277	96.8	0.7499	1.7372	0.9873	98.1
S_3	α_1	1.0232	1.9784	0.9552	96.5	0.8473	1.3661	0.5188	96.3
	α_2	0.9544	2.0739	1.1195	95.9	0.8480	1.4075	0.5595	96.4
	β_{11}	0.1499	0.4173	0.2674	95.4	0.3271	0.6082	0.2811	96.6
	β_{12}	0.1370	0.4306	0.2936	95.1	0.2829	0.6150	0.3321	95.4
	β_{21}	0.1074	1.1263	1.0188	95.6	0.7660	1.6923	0.9262	97.3
	β_{22}	0.0089	1.2450	1.2361	95.8	0.7424	1.6981	0.9557	97.4
S_4	α_1	0.9797	2.0444	1.0647	97.0	0.8904	1.4378	0.5474	96.9
	α_2	0.9608	2.1021	1.1413	96.3	0.8650	1.3969	0.5319	95.6
	β_{11}	0.1473	0.4145	0.2672	96.2	0.3029	0.5889	0.2859	96.3
	β_{12}	0.1324	0.4264	0.2940	96.3	0.3035	0.6054	0.3018	96.6
	β_{21}	0.0389	1.1924	1.1536	96.2	0.7172	1.6831	0.9659	97.9
	β_{22}	0.0000	1.2874	1.2874	96.4	0.7393	1.6969	0.9576	97.6
S_5	α_1	1.0698	1.9198	0.8499	97.3	0.7863	1.3387	0.5524	96.3
	α_2	1.0268	1.9773	0.9504	96.4	0.7392	1.3502	0.6109	96.1
	β_{11}	0.1681	0.4051	0.2370	96.5	0.3328	0.7187	0.3859	96.0

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
	β_{12}	0.1348	0.4411	0.3063	96.8	0.3326	0.7255	0.3929	96.5
	β_{21}	0.2017	0.9353	0.7336	96.0	0.7365	1.7457	1.0091	96.7
	β_{22}	0.1236	1.0253	0.9016	97.1	0.7639	1.6940	0.9301	96.9

Table 14. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for NH distribution at $n = 100$ and $m = 60$

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	α_1	0.9581	2.0629	1.1047	96.6	0.8352	1.3717	0.5365	95.8
	α_2	0.9508	2.0917	1.1409	96.4	0.8871	1.4780	0.5908	97.5
	β_{11}	0.1405	0.4268	0.2863	94.9	0.3007	0.6150	0.3143	96.1
	β_{12}	0.1353	0.4299	0.2946	94.1	0.2949	0.6142	0.3192	96.4
	β_{21}	0.0000	1.3419	1.3419	96.2	0.7738	1.8401	1.0663	97.8
	β_{22}	0.0000	1.2802	1.2802	95.1	0.7463	1.7619	1.0156	98.0
S_2	α_1	1.0340	1.9546	0.9206	97.4	0.7579	1.3757	0.6179	96.9
	α_2	0.9782	2.0418	1.0636	96.7	0.7832	1.3977	0.6145	97.5
	β_{11}	0.1638	0.4088	0.2450	96.2	0.3294	0.7123	0.3830	95.8
	β_{12}	0.1433	0.4291	0.2858	96.6	0.3346	0.7331	0.3985	96.9
	β_{21}	0.1512	0.9988	0.8475	96.7	0.7102	1.7286	1.0184	96.4
	β_{22}	0.0884	1.0684	0.9800	96.9	0.7510	1.7219	0.9708	98.0
S_3	α_1	0.9413	2.0751	1.1338	96.4	0.8592	1.4087	0.5495	96.7
	α_2	0.8820	2.1467	1.2648	95.9	0.8535	1.4337	0.5801	96.4
	β_{11}	0.1456	0.4164	0.2708	95.4	0.3137	0.6043	0.2906	97.5
	β_{12}	0.1388	0.4250	0.2862	95.9	0.3051	0.5877	0.2826	97.1
	β_{21}	0.0779	1.1853	1.1074	95.1	0.7531	1.6888	0.9357	97.6
	β_{22}	0.0204	1.2588	1.2384	95.7	0.7500	1.7044	0.9544	98.0
S_4	α_1	0.9695	2.0515	1.0820	96.5	0.8551	1.4260	0.5709	96.0
	α_2	0.9150	2.1136	1.1986	96.0	0.8687	1.3887	0.5199	97.2
	β_{11}	0.1446	0.4248	0.2803	95.6	0.3090	0.5935	0.2846	96.8
	β_{12}	0.1381	0.4338	0.2957	95.1	0.3116	0.6016	0.2900	95.7
	β_{21}	0.0344	1.2142	1.1798	96.0	0.6549	1.6902	1.0354	96.1
	β_{22}	0.0000	1.2828	1.2828	96.6	0.7206	1.6900	0.9694	96.4
S_5	α_1	1.0953	1.8983	0.8029	95.2	0.7194	1.3198	0.6003	96.0
	α_2	0.9670	2.1264	1.1595	96.8	0.7236	1.3356	0.6121	95.8
	β_{11}	0.1723	0.3951	0.2228	96.3	0.3615	0.7569	0.3953	97.7
	β_{12}	0.1529	0.4019	0.2490	97.6	0.3313	0.7304	0.3991	95.8
	β_{21}	0.2317	0.9088	0.6771	97.0	0.7270	1.8336	1.1066	97.2
	β_{22}	0.1521	0.9730	0.8209	96.2	0.7162	1.6959	0.9798	95.9

Table 15. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for NH distribution at $n = 100$ and $m = 80$

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	α_1	1.0014	1.9767	0.9753	96.5	0.8672	1.4557	0.5885	96.1
	α_2	0.9671	2.0335	1.0663	96.5	0.8673	1.4477	0.5804	96.0
	β_{11}	0.1370	0.4343	0.2973	95.5	0.2891	0.5819	0.2928	96.5
	β_{12}	0.1385	0.4297	0.2912	95.6	0.3136	0.5886	0.2749	98.2
	β_{21}	0.0000	1.2750	1.2750	95.6	0.6209	1.6470	1.0261	95.3

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
	β_{22}	0.0000	1.2940	1.2940	96.4	0.6088	1.6615	1.0528	96.3
S_2	α_1	1.0260	1.9611	0.9351	97.8	0.8092	1.4226	0.6134	96.6
	α_2	0.9928	2.0285	1.0357	96.9	0.7858	1.4112	0.6253	95.9
	β_{11}	0.1570	0.4077	0.2506	96.2	0.3278	0.6729	0.3450	98.2
	β_{12}	0.1391	0.4231	0.2840	95.9	0.3121	0.6344	0.3223	96.9
	β_{21}	0.0896	1.0831	0.9935	95.3	0.6785	1.6941	1.0156	96.6
	β_{22}	0.0072	1.1713	1.1641	96.2	0.6988	1.7464	1.0475	96.9
S_3	α_1	0.9434	2.0292	1.0857	96.4	0.8707	1.4873	0.6166	96.3
	α_2	0.9215	2.0581	1.1366	96.7	0.8537	1.4594	0.6058	95.9
	β_{11}	0.1437	0.4297	0.2860	95.5	0.3024	0.5890	0.2865	98.2
	β_{12}	0.1322	0.4443	0.3121	95.4	0.3068	0.5840	0.2773	97.3
	β_{21}	0.0000	1.3405	1.3405	96.1	0.5905	1.6456	1.0550	96.2
	β_{22}	0.0000	1.4175	1.4175	96.1	0.6994	1.7248	1.0254	98.2
S_4	α_1	0.9951	2.0382	1.0431	96.7	0.8963	1.4636	0.5673	96.9
	α_2	0.9725	2.0632	1.0907	96.5	0.9054	1.4761	0.5707	97.2
	β_{11}	0.1580	0.3915	0.2335	95.8	0.2887	0.5656	0.2769	96.0
	β_{12}	0.1407	0.4135	0.2728	95.5	0.3019	0.5778	0.2759	97.0
	β_{21}	0.0756	1.1268	1.0512	96.1	0.6619	1.6451	0.9832	96.0
	β_{22}	0.0000	1.2574	1.2574	96.8	0.7127	1.7442	1.0315	98.3
S_5	α_1	1.0603	1.9240	0.8637	97.2	0.7974	1.4596	0.6622	97.0
	α_2	1.0173	2.0065	0.9891	96.3	0.8205	1.3699	0.5494	97.4
	β_{11}	0.1660	0.4001	0.2341	95.9	0.3001	0.6537	0.3536	96.4
	β_{12}	0.1493	0.4135	0.2642	95.9	0.3320	0.6646	0.3326	96.6
	β_{21}	0.1697	0.9853	0.8157	96.6	0.6779	1.6527	0.9748	96.3
	β_{22}	0.0591	1.1036	1.0445	97.0	0.7306	1.6954	0.9648	97.1

Table 16. Interval estimates, AILs and CP(%) values of the ML and BE using MCMC for complete sampling step-stress for NH distribution at $n = 60, 80$ and 100

n	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
60	α_1	1.0049	2.0198	1.0149	96.9	0.8616	1.4422	0.5805	96.0
	α_2	0.9277	2.1227	1.1950	97.5	0.8508	1.4173	0.5664	96.1
	β_{11}	0.1411	0.4203	0.2792	95.1	0.3303	0.6326	0.3023	98.6
	β_{12}	0.1311	0.4305	0.2994	95.4	0.2952	0.6004	0.3051	96.1
	β_{21}	0.0383	1.1953	1.1570	95.1	0.6446	1.6218	0.9772	95.8
	β_{22}	0.0000	1.3433	1.3433	96.1	0.7458	1.7580	1.0122	98.1
S_2	α_1	1.0063	1.9930	0.9867	96.6	0.8337	1.4171	0.5834	95.2
	α_2	0.9359	2.0793	1.1433	96.4	0.8462	1.4543	0.6081	95.8
	β_{11}	0.1560	0.4069	0.2508	95.7	0.2855	0.5907	0.3052	96.3
	β_{12}	0.1410	0.4253	0.2843	94.9	0.3037	0.5897	0.2861	96.8
	β_{21}	0.1043	1.0982	0.9939	95.1	0.6750	1.6353	0.9603	97.4
	β_{22}	0.0270	1.2010	1.1740	94.8	0.5959	1.6154	1.0195	95.8
S_3	α_1	1.0603	1.9240	0.8637	97.2	0.7974	1.4596	0.6622	97.0
	α_2	1.0173	2.0065	0.9891	96.3	0.8205	1.3699	0.5494	97.4
	β_{11}	0.1660	0.4001	0.2341	95.9	0.3001	0.6537	0.3536	96.4
	β_{12}	0.1493	0.4135	0.2642	95.9	0.3320	0.6646	0.3326	96.6
	β_{21}	0.1697	0.9853	0.8157	96.6	0.6779	1.6527	0.9748	96.3
	β_{22}	0.0591	1.1036	1.0445	97.0	0.7306	1.6954	0.9648	97.1

Table 17. Est. and St.Ers of the ML and BE using MCMC for illustrative example

Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX	
	Est.	St.Er	Est.	St.Er	Est.: $\nu = -0.5$	Est.:$\nu = 0.5$
β_{11}	1.7862	2.2538	0.8199	0.2420	0.8057	0.8350
β_{12}	1.7121	2.8335	4.2366	2.9518	2.9924	7.2274
β_{21}	1.1683	1.6557	3.1935	1.5610	2.7082	3.9038
β_{22}	1.2362	2.3203	0.6495	0.3658	0.6175	0.6843
α_1	1.0885	1.9117	4.4086	1.9956	3.6670	5.6325
α_2	1.1502	2.6598	0.6431	0.4759	0.5895	0.7023

Table 18. Interval estimates and ILs values of the ML and BE using MCMC for an illustrative example

Parm.	Asy-CI			HPD		
	Lower	Upper	AIL	Lower	Upper	AIL
β_{11}	0.0000	6.2035	6.2035	0.4279	1.2680	0.8402
β_{12}	0.0000	7.2657	7.2657	1.1530	11.0140	9.8610
β_{21}	0.0000	4.4133	4.4133	0.9345	6.2524	5.3178
β_{22}	0.0000	5.7840	5.7840	0.1331	1.3129	1.1798
α_1	0.0000	4.8354	4.8354	1.4500	8.4061	6.9561
α_2	0.0000	6.3633	6.3633	0.0673	1.5143	1.4470

8. Conclusion

In this paper, we studied a simple step-stress ALT model for the NH distribution under progressive Type-II censored data in the presence of competing causes of risks. We have derived the MLEs, and asymptotic confidence interval estimates for the unknown parameters. Also, we computed BEs and the corresponding HPD interval estimates under informative priors based on two different types of loss functions LINEX and squared error loss functions. We then conducted a simulation study to assess the performance of all these procedures and an explanatory instance has been offered to demonstrate all the methods of inference developed. The calculations have been made based on different sample sizes and five different progressive censoring schemes.

Finally, the simulation results demonstrate that both the MLEs and Bayesian estimates become better in terms of MSEs as the number of failures increases for fixed n. The MLEs and BEs for fixed sample size and pre-fixed number of failures increase, and the MSEs and Average estimate for unknown parameters decrease. The MLEs for scheme 5 have smaller MSEs than the other four schemes. Also, the MLEs for scheme 3 have the largest MSEs. The BEs of shape parameters $\alpha_1 and\ \alpha_2$ under LINEX loss function $(\nu = 0.5)$ have the smallest MSEs compared to estimates under the SE loss function, LINEX loss function $(\nu = -0.5)$). The BEs of scale parameters $\beta_{11}, \beta_{12}, \beta_{21}$ and β_{22} under LINEX loss function $(\nu = -0.5)$ have the smallest MSEs as compared with estimates under the SE loss function and LINEX loss function $(\nu = 0.5)$ From the perspective of the confidence/credible intervals, the average lengths of all CIs and HPD credible intervals become smaller as m increases.

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Reference

Balakrishnan, N. and Aggarwal, R. (2000). *Progressive Censoring: Theory, Methods and Applications*. Springer, Science & Business Media. https://doi.org/10.1007/978-1-4612-1334-5

Balakrishnan, N. and Cramer, E. (2014). The art of progressive censoring. Statistics for industry and technology. New York: Birkhauser. http://doi.org/ 10.1007/978-0-8176-4807-7

Balakrishnan, N. and Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type-II censoring. *Journal of statistical planning and inference*, 138(12): 4172–4186. https://doi.org/10.1016/j.jspi.2008.03.036.

Balakrishnan, N. and Sandhu, R. (1995). A simple simulation algorithm for generating progressive Type-II censored samples. *The American Statistician*, 49(2): 229–230. https://doi.org/10.1080/00031305.1995.10476150.

Chen, M.-H. and Shao, Q.-M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics*, 8(1): 69–92. https://doi.org/10.1080/10618600.1999.10474802.

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, 5(3): 327–339. https://doi.org/10.1080/00401706.1963.10490102.

Cohen, A. C. (1965). Maximum Likelihood Estimation in the Weibull Distribution Based On Complete and On Censored Samples. *Technimetrics*, *7*(4): 579–588. https://doi.org/10.1080/00401706.1965.10490300.

Crowder, M. (1991). On the identifiability crisis in competing risks analysis. *Scandinavian Journal of Statistics*, 18(3): 223–233. http://www.jstor.org/stable/4616205.

Crowder, M. J. (2001). *Classical Competing Risks*. Chapman and Hall/CRC.

David, H. A. and Moesch Berger, M. L. (1978). *The Theory of Competing Risks*. London: Griffin's Statistical Monographs and Courses.

El-Shahat, M., Abu-El Azm, W. and Abd El-Aziz, Y.S. (2025). Inference on the accelerated competing failure model from the Rayleigh distribution under Type-I censored data. *Pakistan Journal of Statistics*, 41(1):51–74.

Epstein, B. (1954). Truncated life tests in the exponential case. *The Annals of Mathematical Statistics*, 25(3): 555–564. http://www.jstor.org/stable/2236837.

Han, D. and Balakrishnan, N. (2010). Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. *Computational Statistics & Data Analysis*, 54(9):2066–2081. https://doi.org/10.1016/j.csda.2010.03.015.

Han, D. and Kundu, D. (2014). Inference for a step-stress model with competing risks for failure from the generalized exponential distribution under Type-I censoring. *IEEE Transactions on Reliability*, 64(1):31–43. https://doi.org/10.1109/TR.2014.2336392.

Kalbfleisch, J. D. and Prentice, R. L. (2011). *Statistical Analysis of Failure Time Data*. John Wiley & Sons. http://doi.org/ 10.1002/9781118032985

Liu, F. and Shi, Y. (2017). Inference for a simple step-stress model with progressively censored competing risks data from Weibull distribution. *Communications in Statistics-Theory and Methods*, 46(14):7238–7255. https://doi.org/10.1080/03610926.2016.1147585.

Nadarajah, S. and Haghighi, F. (2011). An extension of the exponential distribution. *Statistics*, 45(6):543–558. https://doi.org/10.1080/02331881003678678.

Nassar, M. and Eissa, F. H. (2005). Bayesian estimation for the exponentiated Weibull model. *Communications in Statistics-Theory and Methods*, 33(10): 2343–2362. https://doi.org/10.1081/STA-200031447.

Nelson, W. (1990). Accelerated Testing: Statistical Models, Test Plans and Data Analyses. Wiley, New York. http://dx.doi.org/10.1002/9780470316795

Robert, C. P., Casella, G., and Casella, G. (1999). *Monte Carlo Statistical Methods*, volume 2. Springer.

Sarhan, A. M., Hamilton, D. C., and Smith, B. (2010). Statistical analysis of competing risks models. *Reliability Engineering & System Safety*, 95(9):953–962. https://doi.org/10.1016/j.ress.2010.04.006.

Shi, Y. M., Jin, L., Wei, C., and Yue, H. B. (2013). Constant-stress accelerated life test with competing risks under progressive Type-II hybrid censoring. *Advanced Materials Research*, 712:2080–2083. https://doi.org/10.4028/www.scientific.net/AMR.712-715.2080.

Singh, S., Singh, U., Kumar, M., and Vishwakarma, P. (2014). Classical and Bayesian inference for an extension of the exponential distribution under progressive Type-II censored data with binomial removals. *Journal of Statistics Applications & Probability Letters*, 1(3):75–86. http://dx.doi.org/10.12785/jsapl/010304.

Upadhyay, S. and Gupta, A. (2010). A bayes analysis of Modified Weibull distribution via Markov Chain Monte Carlo simulation. *Journal of Statistical Computation and Simulation*, 80(3):241–254. https://doi.org/10.1080/00949650802600730.

Wu, M., Shi, Y., and Sun, Y. (2014). Inference for accelerated competing failure models from Weibull distribution under Type-I progressive hybrid censoring. *Journal of Computational and Applied Mathematics*, 263:423–431. https://doi.org/10.1016/j.cam.2013.12.048.

Xu, A. and Tang, Y. (2011). Objective Bayesian analysis of accelerated competing failure models under Type-I censoring. *Computational Statistics & Data Analysis*,55(10):2830–2839. https://doi.org/10.1016/j.csda.2011.04.009.