ON PERT/COST PROGRAMMING MODELS

By

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PERT/COST analysis is an important extension of the PERT/TIME technique. In some cases, decreasing the time needed to perform each activity, and therefore to accomplish the whole project, could be acheived by increasing the direct costs [(3)] related to those activities. PERT/COST analysis is mainly directed to determining the minimum direct cost in addition to the time needed for performing each activity, and consequently for performing the whole project.

Various techniques have been developed to takle such cases if the relation between time and cost for each activity is assumed to take a linear form. Under this assumption of linearity, linear programming seems to be the most applicable technique specially in large projects.

After presenting a linear programming model for deciding the activity times that yeild the minimum cost required to accomplish the project within a given time ; this paper introduces a general non-linear programming model that could be used to determine the time which should be assigned to each activity, together with the related cost, even if the relation between time and cost is nonlinear. It is also shown that the linearity assumption gives a special case of the non-linear model.

A Linear Programming Model for PERT/COST analysis :

Consider a network consisting of n activities, and suppose that the time and cost information related to the j^{th} activity ($j = 1 , \ldots, n$) are given by :

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(3) The total cost for any activity-and therefore for the whole project-is the sum direct: and indircet costs. In PERT/COST analysis costs usually mean the direct costs, which includet wages, depretiation of capital., costs of machine renting. The indirect costs such as : los profits, penalty costs for not meating the scheduel, opportunity cost, ... are usually not included in the PERT/COST analysis.

T_{N_j}, the «normal time» during which the j^{th} activity might be performed,

C_{N_j}, the «normal cost», i.e., the cost corresponding to the normal time of the J^{th} activity.

T_{C_j}, the «crash time», or the least time during which the J^{th} activity might be performed, $(T_{C_j} \leqslant T_{N_j})$,

C_{C_j}, the «crash cost», i.e., the cost corresponding to the crash time of the J^{th} activity, $(C_{C_j} \leqslant C_{N_j})$.

Then, under the linearity assumption, the relation between time and cost for the J^{th} activity is represented by a straight line as follows :

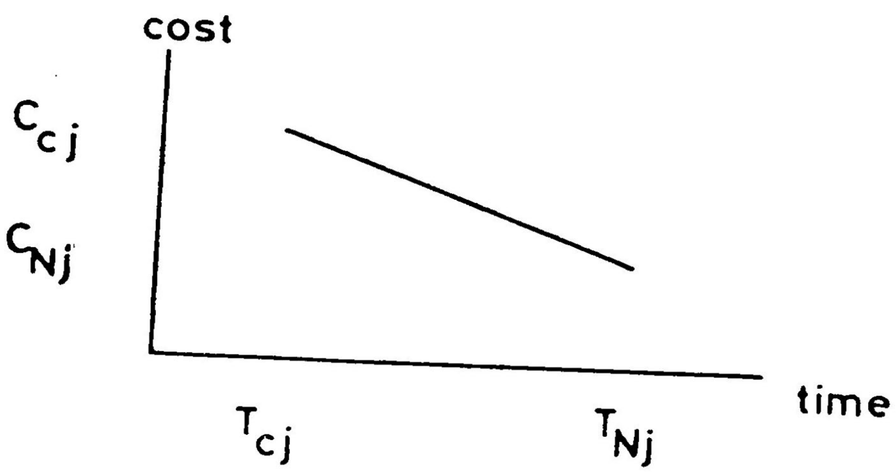


Fig. 1

Now, if we define the variables x_j $(j = 1, \ldots, n)$ as the lengths of time in excess of the crash time for the J^{th} activity, then each x_j should not exceed the difference between the crash and normal times of activity j, and, according to the linearity assumption, the cost of performing the J^{th} activity whose crash time-increases by x_j units is given by :

$$C_{C_j} - R_j \; x_j$$

where R_j is the absolute value of the slope of the cost line, i.e.,

$$R_j = \frac{C_{C_j} - C_{N_j}}{T_{N_j} - T_{C_j}}$$

Let $P_1 \ldots P_k$ denote all the possible paths between the first and last events in the network, and let the parameter V denote the total time given to the project. Then V should be greater than or equal to each possible path length.

Therefore the problem is to assign the time and cost that each activity needs in order to accomplish the project within V units of time and at minimum cost.

The linear programming model for this problem is :

Find the values of x_j, s (j = 1, . . ., n) which minimize :

$$\sum_{j=1}^{N} (C_{cj} - R_j \; x_j)$$

subject to :

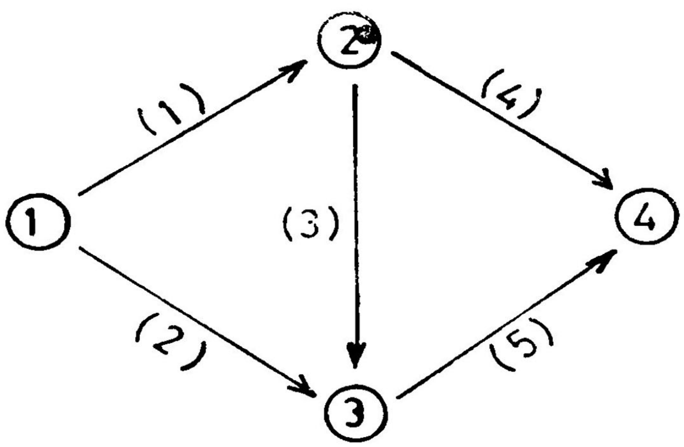
$$\sum_{j \in P_r} (T_{cj} + x_j) \leqslant V \qquad\qquad r = 1, \ldots, k$$

and
$$x_j \leqslant T_{Nj} - T_{Cj} \quad j = 1, \ldots, n$$
$$x_j \geqslant 0 \qquad\qquad j = 1, \ldots, n$$

The solution of this model yeilds the time in excess of the crash time for each activity. The optimal value of the objective function represents the minimum cost required to accomplish the project within the given time V If V is permitted to vary, we have a parametric linear programming.

For illustration we present the following example :

Consider the project which is represented by the following network, and which consists of five activities whose normal and crash times, and normal and crash costs are given as follows :



Activity j	Normal time T_{N_j}	Normal Cost C_{N_j}	Crash time T_{C_j}	Crash cots C_{C_j}
1	9	10	6	31
2	17	2	9	10
3	7	15	6	25
4	5	10	3	14
5	3	10	1	16

If it is required to accomplish the project within 16 units of time at minimm cost, what should be the time and cost of each activity, and what should be the critical path for this nework ?

The possible paths in this problem are :

P_1 including activities 1 and 4.

P_2 including activities 1, 3, and 5.

P_3 including activities 2 and 5.

Thus, the linear programming nodel is :

To find $x_1, \ldots, x_5$ which minimize :

$$96 - 7x_1 - x_2 - 10x_3 - 2x_4 - 3x_5$$

Subject to :

$$(6 + x_1) + (3 + x_4) \leqslant 16$$

$$(6 + x_1) + (6 + x_3) + (1 + x_5) \leqslant 16$$

$$(9 + x_2) + (1 + x_5) \leqslant 16$$

$$x_1 \leqslant 3$$

$$x_2 \leqslant 8$$

$$x_3 \leqslant 1$$

$$x_4 \leqslant 2$$

$$x_5 \leqslant 2$$

where $x_j \geqslant 0$, j 1, ..., n.

Applying the simplex method, the solution for this linear programming problem is given by :

$$x_1 = 2,\ x_2 = 6,\ x_3 = 1,\ x_4 = 2,\ x_5 = 0,$$

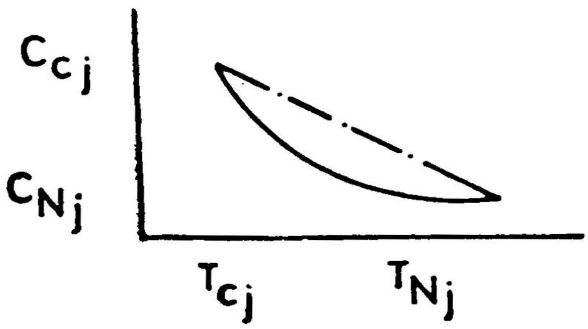
and the value of the objective function is 62 units. In other werds, the optimal time and cost for each activity are :

Activity	Time	Cost
1	8	17
2	15	4
3	7	15
4	5	10
5	1	16

Applying the PERT/Time procedure using this activity time information, we find that there are two critical paths : P2 and P3. consequently the critical activities are (1, 3 and 5) or (2 and 5). The project time is 16 units of time and its cost is 62 cost units.

A Non-linear programming Mode for PERT/COST analysis :

Although it simplifies the problem, the linearily assumption might lead to inadequate results if the relation between time and costs is far from linearity for some activities, specially if the difference between normal and crasah times is large :



In what follows the paper introduces a non-linear programming model that does not depend on the linearity assumption':

Consider a network that consists of n activities, and let the cost function $f_j(T_j)$ represent the relation between the time Tj of performing the j^{th} activity and the cost of that activity, where the time variable Tj ranges over the interval (T_{C_j}, T_{N_j}) and where $j = 1, \ldots n$. Then the problen is to assigne the time and cost for performing each activity so as to accomplish the project within V units of time and at minimm costs.

Therefore, the total cost of the project, $\sum_{j=1}^{n} f_j(T_j)$, represents the objective function which is to be minimized subject to the following constraints :

1. Each path time should be less than or equal to the given preject time V.

2. Each j^{th} activity time should be greater than or equal to the lower time limit T_{C_j}.

3. Each j^{th} activity time should be less than or equal to the upper time limit T_{N_j}.

The non-linear programming model for this problem is :

Find the value of T_j, s (j = 1, 1 . . . , n) which minimize :

$$\sum_{j=1}^{n} f_j (T_j)$$

Sbuject to
$$\begin{array}{llll} \sum_{j \varepsilon P_r} T_j & \leqslant & V & r = 1, \ldots, k. \\ T_j & \geqslant & T_{C_j} & j = 1, \ldots, n. \\ T_j & \leqslant & T_{N_j} & j = 1, \ldots n. \\ T_j & \geqslant & 0 & j = 1, \ldots n. \end{array}$$

It could easily be shown that the linear programming model presented above is a specinal case of this goneral non-linear model :

Since, by definition, $xj = Tj - T_{C_j}$ or equivelantly

$Tj = T_{C_i} + xj$ (j = 1, 1, . . . n),

then :

the objective function for the linear programming model is given by :

$$\sum_{j=1}^{n} f_j (T_j) = \sum_{j=1}^{n} (C_{c_j} - R_j (T_j - T_{c_j}))$$

and the constraints are given by :

$$\sum_{j \in P_r} T_j \leqslant V \quad \text{implies that} \quad \sum_{j \in P_r} (T_{c_j} + X_{k_j}) \leqslant V$$

$$T_j \leqslant T_N \text{ implies that } X_j + T_{Cj} \leqslant T_{Nj}$$

$$\text{i.e.} \quad x_j \leqslant T_{Nj} - T_{Cj} \quad \text{for all } j$$

$$T_i \geqslant T_{Cj} \text{ implies that } x_j + T_{Cj} \geqslant T_{Cj}$$

$$\text{i.e.} \quad x_j \geqslant 0 \quad \text{for all } j$$

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