doi : 10.21608/esju.2025.351499.1063 Research Article

The Egyptian Statistical Journal (ESJ), 69 (1): 127-146

Homepage: https://esju.journals.ekb.eg/

The Egyptian Statistical Journal

Print ISSN 0542-1748– Online ISSN 2786-0086

Statistical Inference and Prediction for Multiply-Hybrid Censored Data with Applications

Mahmoud A. Eltehiwy[1*], Mohamed A. Hamouda[2], Noura A. T[1]

Received 08 January 2025; revised 20 April 2025; accepted 30 April 2025; online 24 May 2025

Keywords

Life-Testing Experiments, power Lindley distribution, Statistical inference, Multiply hybrid censoring, Two-Sample prediction.

Abstract

This study aims to enhance the prediction of failure times for specific units in lifetime experiments where complete observation of all failure times is impractical, introducing a novel approach to handling multiply-hybrid censored data. Leveraging the power of the Lindley distribution—recognized for its adaptability to diverse real-life datasets—the research develops statistical inference techniques to estimate distribution parameters with high precision and implements a two-sample prediction method to forecast unobserved failure times. Principal findings demonstrate that both maximum Likelihood and Bayesian estimators, supported by Markov Chain Monte Carlo methods, yield accurate parameter estimates, with Bayesian approaches showing slight superiority. Simulation results reveal reduced mean square errors and narrower credible intervals as sample sizes increase, while real-life applications to aircraft failure and leukaemia survival data confirm the power of the Lindley distribution's excellent fit. These results signify a robust framework for improving prediction accuracy under data constraints, offering significant advancements in reliability analysis and survival modelling. By providing a versatile methodology validated across industrial and clinical contexts, this study impacts statistical practice by equipping researchers with tools to address incomplete data challenges effectively, with broad implications for life-testing experiments in engineering, medicine, and beyond.

Mathematical Subject Classification: 62F15, 62F25, 62N02, 62N05

1. Introduction

In reliability engineering and survival analysis, the accurate estimation of lifetime distributions is critical for predicting product failure rates and optimizing maintenance schedules. Censoring schemes are widely employed in such studies to balance the trade-off between data collection costs and the precision of parameter estimates. Among these schemes, hybrid censoring has gained significant attention due to its ability to combine the advantages of both Type-I and Type-II censoring, ensuring time efficiency and providing enough failure observations.

1.1 Censoring Schemes in Life-Testing

Censoring schemes are categorized based on how the experiment is concluded. In Type-I censoring**,** the experiment ends after a predetermined time (T). This approach ensures a fixed test duration but

✉ Corresponding author*: Mahmoud.abdelmone@eps.bsu.edu.eg.
[1,3] Faculty of the Politics and Economics, Beni-Suef University, Beni-Suef, Egypt.
[2] Faculty of the Commerce, Aswan University, Aswan, Egypt.

may overlook the efficiency of the components being tested. Conversely, Type-II censoring concludes the experiment after a predetermined number of components (r) have been tested. This method prioritizes maintaining efficiency over the fixed duration of the test, with the test duration determined randomly.

To address the limitations of Type-I and Type-II censoring, hybrid censoring schemes combine the strengths of both methods, offering greater flexibility and manageability. These schemes rely on two key parameters: r, the number of failures to be observed, and T, the maximum allowable test duration. There are two primary hybrid censoring schemes. The first is the Type-I hybrid censoring scheme**,** in which the test concludes at $T_1 = min\{X_{r,n}, T\}$, where $X_{r,n}$ represents the r^{th} ordered failure time among n tested items. The second is the Type-II hybrid censoring scheme**,** where the test ends at $T_2 = max\{X_{r,n}, T\}$.

Research on hybrid censoring methods has evolved significantly over time, with numerous studies focusing on parameter estimation for various distribution functions. Childs et al. (2003) were among the first to explore exact likelihood inference under Type-I and Type-II hybrid censoring for the exponential distribution. Banerjee and Kundu (2008) later extended these techniques to the Weibull distribution, demonstrating their effectiveness in statistical inference. Ganguly et al. (2012) contributed by developing inference methods for the two-parameter exponential distribution, while Dey and Pradhan (2014) applied hybrid censoring approaches to the generalized inverted exponential distribution. Al-Zahrani and Gindwan (2014) further investigated parameter estimation under hybrid censoring for the Lindley distribution. Kohansal et al. (2015) explored Type-II hybrid censoring for the weighted exponential distribution, thereby expanding the scope of hybrid censoring applications.

To address challenges associated with losses due to failures—denoted by R_i—that occur between consecutive observations without precise failure times for all affected units, it became necessary to develop a new scheme to improve the effectiveness of existing censoring techniques. Lee and Lee (2018) introduced a novel approach known as the multiply Type-II hybrid censoring system, which provided a unique solution to account for such losses and enhance the efficiency of parameter estimation methods. Further advancements in hybrid censoring methods have been made in recent years. Jeon and Kang (2020) investigated parameter estimation for two distinct distributions under the multiply Type-II hybrid censoring framework, offering additional insights into its applications and effectiveness in statistical inference. In one study, they applied Bayesian estimation techniques to the exponential distribution using a generalized multiply Type-II hybrid censoring scheme. In another, they extended their analysis to the half-logistic distribution, demonstrating the versatility of this method in parameter estimation. Mansour and Aboshady (2023) focused on predicting failure times for unobserved events using real-world data, further highlighting the practical relevance of hybrid censoring techniques. Collectively, these studies underscore the growing importance of hybrid censoring schemes in enhancing estimation accuracy and reliability in real-world applications.

1.2 Power Lindley Distribution

Researchers have extensively studied the power Lindley (PL) distribution due to its flexibility in modeling lifetime data. Ghitany et al. (2013) proposed the PL distribution, which is indexed by both shape and scale parameters. The Lindley distribution family is well-suited to modeling real phenomena, as it does not assume a constant hazard rate—an assumption that rarely holds in real-life systems with time-independent failure rates. The PL distribution can exhibit both decreasing and increasing hazard rates, as well as unimodal distribution functions. Such behaviors are commonly observed in life-testing experiments. Additionally, Singh et al. (2014) presented a study dealing with the classical and Bayesian estimation of the hybrid censored lifetime data under the assumption that the lifetimes follow the power Lindley distribution.

Despite these advancements, limited research has focused on parameter estimation for the power Lindley distribution under multiply Type-II hybrid censoring. This censoring scheme, which permits multiple stages of data collection, offers a unique balance between efficiency and reliability, making it particularly suitable for real-world applications where test duration and cost are critical constraints. By addressing this gap, the present study contributes to the expanding body of research on hybrid censoring and lifetime data analysis.

The corresponding cumulative distribution function (cdf) and probability density function (pdf) of the PL distribution are provided to facilitate further analysis as follows:

$$G_{PL}(x) = 1 - \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right) e^{-\beta x^{\alpha}}, \quad x > 0, \alpha > 0, \beta > 0, \tag{1}$$

and

$$g_{PL}(x; \alpha, \beta) = \frac{\alpha \beta^2}{\beta + 1}\left(1 + x^{\alpha}\right) x^{\alpha - 1} e^{-\beta x^{\alpha}}, \quad x > 0, \alpha > 0, \beta > 0, \tag{2}$$

respectively.

1.3 Objectives and Contributions

The primary challenge addressed in this study is the estimation of parameters for the power Lindley distribution under a multiply Type-II hybrid censoring scheme. Traditional methods often struggle to provide accurate estimates in scenarios involving limited data or complex censoring structures. To overcome this limitation, we propose a comprehensive framework that integrates both Bayesian and frequentist approaches, leveraging the strengths of each to improve estimation accuracy.

The methodology involves:

1. Deriving maximum likelihood estimators (MLEs) for the power Lindley distribution parameters under multiply Type-II hybrid censoring.
2. Developing Bayesian estimators using non-informative priors to ensure objectivity in the absence of prior information.
3. Comparing the performance of these estimators through extensive simulation studies and real-world applications.

The objectives of this study are:

1. To provide efficient and reliable parameter estimation methods for the power Lindley distribution under multiply Type-II hybrid censoring.
2. To demonstrate the advantages of the proposed scheme over traditional censoring methods in terms of estimation accuracy and test efficiency.
3. To highlight the practical applicability of the proposed framework in real-life reliability testing scenarios.

The structure of this study is as follows: Section 2 discusses the maximum likelihood estimators (MLEs) and presents approximate confidence intervals (ACIs) for the model parameters. Section 3 focuses on Bayesian estimation techniques and introduces the Markov Chain Monte Carlo (MCMC) method. Section 4 highlights Bayesian prediction research based on effective estimators. Section 5 demonstrates the application of the proposed methods using two real data sets. Section 6 presents simulation studies conducted to evaluate the performance of the developed estimators. Finally, Section 7 concludes the study with a summary of the findings and their implications.

2. Maximum-likelihood estimation

Log-likelihood functions form the foundation for developing parameter estimators based on data. Maximum likelihood estimators offer several advantages, including their ability to meet invariant characteristics. They also possess desirable asymptotic properties, such as being asymptotically unbiased, asymptotically normally distributed, and achieving the lowest variance asymptotically. For further insights into probability theory and related concepts, refer to Azzalini (2017) and Royall (2017).

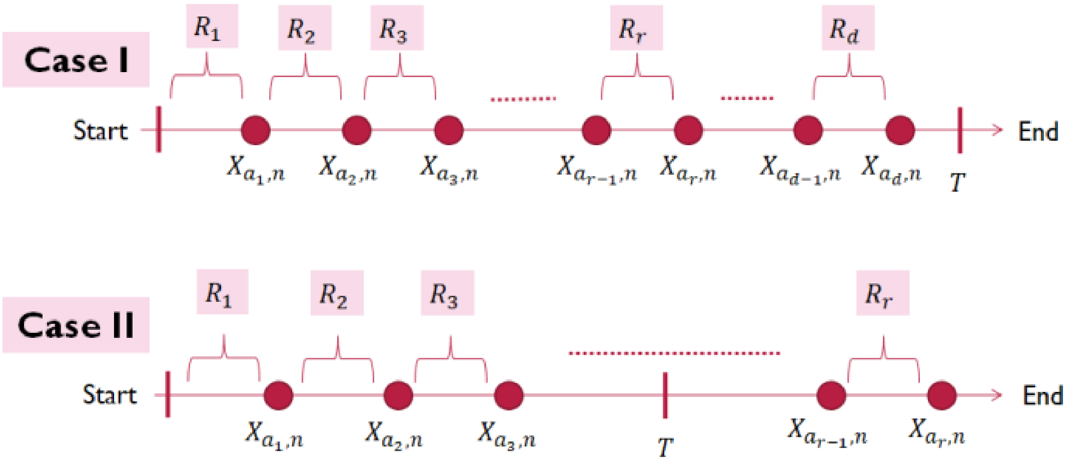

Figure 1. Multiply Type-II hybrid censoring scheme.

Suppose some initial and middle observations are censored as well as some final observations are censored. That is, we only observed

$$X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_r:n},$$

where

$$1 \le a_1 < a_2 < \cdots < a_r \le n,$$

and the others go unnoticed. We may not be aware of the precise times at which certain units fail. The multiply Type-II censoring technique is this one. There are several ways that a multiply Type-II censored sample might arise. It is feasible because of certain experimental or mechanical issues that arise while the devices are being checked and adjusted. When some units fail between two points of observation, with the precise moments of failure going unnoticed, massively censored samples also naturally occur. A generalization of Type-II censoring schemes, the multiply Type-II censoring system only observes the first r failure times. Similarly, the Type-II hybrid censoring technique may censor the start, intermediate, and end dates. Some units may malfunction between two observation points under the Type-II hybrid censoring scheme, although it may not be possible to pinpoint the precise moments at which these units malfunction. There are two kinds of multiply Type-II hybrid censoring schemes:

$$\text{Case I:}\ X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_r:n} < \cdots < X_{a_d:n} < T,$$

$$\text{Case II:}\ X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_d:n} < T < X_{a_{d+1}:n} < \cdots < X_{a_r:n},$$

where $X_{a_i:n}$ denotes a_ith observed failure time, R_i is not exactly an unobserved or lost observation number between $X_{a_{i-1}:n}$ and $X_{a_i:n}$, r is the predetermined observation number, and d is the failure observation number until T. A schematic representation of the multiply Type-II hybrid censoring scheme is presented in Figure 1.

The likelihood function is

$$\text{Case I: } L \propto \prod_{i=1}^{d} f\left(x_{a_i:n}\right) \prod_{i=1}^{d-1} \left[F\left(x_{a_{i+1}:n}\right) - F\left(x_{a_i:n}\right)\right]^{R_{i+1}} \left[F\left(x_{a_1:n}\right)\right]^{R_1} \left[1 - F\left(x_{a_{d:n}}\right)\right]^{n-a_d},$$

$$\text{Case II: } L \propto \prod_{i=1}^{r} f\left(x_{a_i:n}\right) \prod_{i=1}^{r-1} \left[F\left(x_{a_{i+1}:n}\right) - F\left(x_{a_i:n}\right)\right]^{R_{i+1}} \left[F\left(x_{a_1:n}\right)\right]^{R_1} \left[1 - F\left(x_{a_{r:n}}\right)\right]^{n-a_r}.$$

Joining cases I and II, we can rewrite the likelihood function as follows:

$$L \propto \prod_{i=1}^{m} f\left(x_{a_i:n}\right) \prod_{i=1}^{m-1} \left[F\left(x_{a_{i+1}:n}\right) - F\left(x_{a_i:n}\right)\right]^{R_{i+1}} \left[F\left(x_{a_1:n}\right)\right]^{R_1} \left[1 - F\left(x_{a_m:n}\right)\right]^{n-a_m}, \tag{3}$$

where m is the number of failure items until the termination point occurred, $R_i = a_i - a_{i-1} - 1$, $a_0 = 0$.

The log-likelihood function is

$$\ln L \propto \sum_{i=1}^{m} \ln f\left(x_{a_i:n}\right) + \sum_{i=1}^{m-1} R_{i+1} \ln\left[F\left(x_{a_{i+1}:n}\right) - F\left(x_{a_i:n}\right)\right] + R_1 \ln\left[F\left(x_{a_1:n}\right)\right]$$
$$+ \left(n - a_m\right) \ln\left[1 - F\left(x_{a_{m:n}}\right)\right]. \tag{4}$$

Substituting by the cdf in Equation (1) and the pdf in Equation (2) in the log-likelihood function Equation (4), we will get:

$$\ln L \propto \sum_{i=1}^{m} \ln\left(\frac{\alpha\beta^2}{\beta+1}\left(1 + x_{a_i:n}^{\alpha}\right) x_{a_i:n}^{\alpha-1} e^{-\beta x_{a_i:n}^{\alpha}}\right) + \sum_{i=1}^{m-1} R_{i+1} \ln[A]$$
$$+ R_1 \ln[B] + \left(n - a_m\right) \ln\left[\left(1 + \frac{\beta x_{a_m:n}^{\alpha}}{\beta+1}\right) e^{-\beta x_{a_m:n}^{\alpha}}\right], \tag{5}$$

where

$$A = \left[\left(\left(1 + \frac{\beta x_{a_i:n}^{\alpha}}{\beta+1}\right) e^{-\beta x_{a_i:n}^{\alpha}}\right) - \left(\left(1 + \frac{\beta x_{a_{i+1}:n}^{\alpha}}{\beta+1}\right) e^{-\beta x_{a_{i+1}:n}^{\alpha}}\right)\right],$$

and

$$B = \left[1 - \left(1 + \frac{\beta x_{a_1:n}^{\alpha}}{\beta+1}\right) e^{-\beta x_{a_1:n}^{\alpha}}\right].$$

Thus, the likelihood equations for α and β are, respectively:

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{m}\left(\frac{1}{\alpha} + \frac{x_{a_i:n}^{\alpha} \ln x_{a_i:n}}{1 + x_{a_i:n}^{\alpha}} + \ln x_{a_i:n} - \beta x_{a_i:n}^{\alpha} \ln x_{a_i:n}\right) +$$

$$\sum_{i=1}^{m-1}\left(\frac{R_{i+1}\left[\left(\left(1+x^{\alpha}_{a_{i+1}:n}\right)\frac{\beta^{2}x^{\alpha}_{a_{i+1}:n}}{\beta+1}e^{-\beta x^{\alpha}_{a_{i+1}:n}}\ln\left(x_{a_{i+1}:n}\right)\right)-\left(\left(1+x^{\alpha}_{a_i:n}\right)\frac{\beta^{2}x^{\alpha}_{a_i:n}}{\beta+1}e^{-\beta x^{\alpha}_{a_i:n}}\ln\left(x_{a_i:n}\right)\right)\right]}{A}\right)+$$

$$\frac{R_1\left(\left(x^{\alpha}_{a_1:n}+1\right)\frac{\beta^{2}x^{\alpha}_{a_1:n}}{\beta+1}e^{-\beta x^{\alpha}_{a_1:n}}\ln\left(x_{a_1:n}\right)\right)}{B}-\frac{\left(n-a_m\right)\left(\left(1+x^{\alpha}_{a_m:n}\right)\frac{\beta^{2}x^{\alpha}_{a_m:n}}{\beta+1}e^{-\beta x^{\alpha}_{a_m:n}}\ln\left(x_{a_m:n}\right)\right)}{\left[\left(1+\frac{\beta x^{\alpha}_{a_m:n}}{\beta+1}\right)e^{-\beta x^{\alpha}_{a_m:n}}\right]}=0, \tag{6}$$

and

$$\frac{\partial \ln L}{\partial \beta}=\sum_{i=1}^{m}\left(\frac{2}{\beta}-\frac{1}{1+\beta}-x^{\alpha}_{a_i:n}\right)+$$

$$\sum_{i=1}^{m-1}\left(\frac{R_{i+1}\left[\left(\frac{\beta x^{\alpha}_{a_{i+1}:n}}{\left(\beta+1\right)^{2}}e^{-\beta x^{\alpha}_{a_{i+1}:n}}\left(2+\beta+\left(1+\beta\right)x^{\alpha}_{a_{i+1}:n}\right)\right)-\left(\frac{\beta x^{\alpha}_{a_i:n}}{\left(\beta+1\right)^{2}}e^{-\beta x^{\alpha}_{a_i:n}}\left(2+\beta+\left(1+\beta\right)x^{\alpha}_{a_i:n}\right)\right)\right]}{A}\right)+$$

$$\frac{R_1\left(\frac{\beta x^{\alpha}_{a_1:n}}{\left(\beta+1\right)^{2}}e^{-\beta x^{\alpha}_{a_1:n}}\left(2+\beta+\left(1+\beta\right)x^{\alpha}_{a_1:n}\right)\right)}{\left[B\right]}-\frac{\left(n-a_m\right)\left(\frac{\beta x^{\alpha}_{a_m:n}}{\left(\beta+1\right)^{2}}e^{-\beta x^{\alpha}_{a_m:n}}\left(2+\beta+\left(1+\beta\right)x^{\alpha}_{a_m:n}\right)\right)}{\left[\left(1+\frac{\beta x^{\alpha}_{a_m:n}}{\beta+1}\right)e^{-\beta x^{\alpha}_{a_m:n}}\right]}=0. \tag{7}$$

It is preferred to solve the two nonlinear equations (6) and (7) numerically using the Newton Raphson technique to achieve an approximation solution since calculating them simultaneously in the two unknown parameters α and β is too challenging. For further details on the processes involved in the Newton Raphson algorithm, See (EL-Sagheer, 2018). Lastly, we will designate the MLEs for the parameters α and β as $\hat{\alpha}$ and $\hat{\beta}$.

The entries of the inverse matrix of the Fisher information matrix,

$$I_{ij}=E\left\{-\left[\partial^{2}L\left(\Phi\right)/\partial\left(\phi_i\right)\partial\left(\phi_j\right)\right]\right\},\quad i,j=1,2,\qquad \Phi=\left(\phi_1,\phi_2\right)=\left(\alpha,\beta\right),$$

provide the asymptotic variances and covariances of the MLEs, $\hat{\alpha}$ and $\hat{\beta}$. However, obtaining the exact closed forms of these expectations is computationally challenging. To address this issue, the observed Fisher information matrix,

$$\hat{I}_{ij}=E\left\{-\left[\partial^{2}L\left(\Phi\right)/\partial\left(\phi_i\right)\partial\left(\phi_j\right)\right]\right\}_{\hat{\Phi}=\Phi},$$

is used, which eliminates the expectation operator E. This observed matrix facilitates the construction of confidence intervals for the parameters. Notably, the entries of the observed Fisher information matrix are derived as simple second partial derivatives of the log-likelihood function, simplifying its computation:

$$\hat{I}(\alpha,\beta)=\begin{pmatrix} -\frac{\partial^2 L}{\partial\alpha^2} & -\frac{\partial^2 L}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 L}{\partial\beta\partial\alpha} & -\frac{\partial^2 L}{\partial\beta^2} \end{pmatrix}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}. \tag{8}$$

The approximate (or observed) asymptotic variance-covariance matrix for the maximum likelihood estimators, denoted as $\hat{V}$, is obtained by inverting the observed information matrix, $\hat{I}(\alpha,\beta)$. Mathematically, this is expressed as

$$\hat{V}=\hat{I}^{-1}(\alpha,\beta)=\begin{pmatrix} \operatorname{Var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\beta}) \\ \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{Var}(\hat{\beta}) \end{pmatrix}. \tag{9}$$

Under certain regularity conditions, the MLEs $(\hat{\alpha},\hat{\beta})$ are approximately distributed as a multivariate normal distribution with a mean vector (α,β) and a covariance matrix $\boldsymbol{I^{-1}}(\alpha,\beta)$, as detailed in Lawless (2011). Using this result, the $100(1-\gamma)\%$ two-sided confidence intervals for α and β can be calculated. These intervals are determined using $\boldsymbol{z_{\gamma/2}}$, the standard normal distribution's percentile corresponding to a right-tail probability of $\gamma/2$:

$$\hat{\alpha}\pm \boldsymbol{z}_{\gamma/2}\sqrt{\operatorname{Var}(\hat{\alpha})} \text{ and } \hat{\beta}\pm \boldsymbol{z}_{\gamma/2}\sqrt{\operatorname{Var}(\hat{\beta})}. \tag{10}$$

3. Bayes estimation

In this section, Bayesian estimates for the two unknown parameters α and β are derived using the squared error loss function (SEL). The parameters are assumed to be independent and follow Jeffrey's prior distributions, given by $\pi_1(\alpha)=\alpha^{-1}$ and $\pi_2(\beta)=\beta^{-1}$, where $\alpha,\beta>0$. Bayesian estimation is a widely adopted approach in statistical inference, as it minimizes the posterior expected loss, as discussed in Ahmed (2014), Ahmed (2017), Danish et al. (2018), Mahmoud et al. (2022), and Mansour and Ramadan (2020). The joint posterior distribution of the parameters, denoted by $\pi^*(\alpha,\beta\mid data)$, is derived using Bayes' theorem. This posterior distribution, up to proportionality, is obtained by combining the likelihood function (Equation 3) with the prior distributions.

$$\pi^*(\alpha,\beta|\ \text{data})=\frac{\pi_1(\alpha)\pi_2(\beta)L(\alpha,\beta|\ \text{data})}{\int_0^\infty\int_0^\infty \pi_1(\alpha)\pi_2(\beta)L(\alpha,\beta|\ \text{data})\,d\alpha\, d\beta}. \tag{11}$$

The (SEL) function was chosen due to its desirable statistical properties in parameter estimation and prediction. One of the primary reasons for using SEL is its ability to minimize the expected squared difference between the estimated and true parameter values, ensuring an optimal estimator in the Bayesian framework. Additionally, SEL provides symmetric penalization, treating overestimation and underestimation equally, which is particularly useful in applications where deviations in both directions are equally undesirable.

From a computational perspective, the SEL function leads to closed-form expressions for Bayesian estimators in certain distributions, making estimation and inference more straightforward. This analytical simplicity enhances the interpretability and efficiency of the proposed method. Furthermore, Bayesian estimators under SEL perform well in small-sample scenarios, as they incorporate prior information, improving estimation accuracy and reliability. If $\hat{\phi}$ represents the estimator for the parameter ϕ, the SEL function is defined as

$$L\left(\phi,\hat{\phi}\right)=\left(\hat{\phi}-\phi\right)^{2}. \tag{12}$$

This property makes the SEL function particularly useful in statistical estimation. Consequently, for any function involving the parameters α and β, denoted as $\mathrm{g}(\alpha,\beta)$, the Bayes estimate can be derived using the SEL function as

$$\hat{\mathrm{g}}_{BS}\left(\alpha,\beta|\ \text{data}\right)=E_{\alpha,\beta\ \text{data}}\left[\mathrm{g}\left(\alpha,\beta\right)\right], \tag{13}$$

where

$$E_{\alpha,\beta\ \underline{X}}\left[\mathrm{g}\left(\alpha,\beta\right)\right]=\frac{\int_0^\infty\int_0^\infty \mathrm{g}\left(\alpha,\beta\right)\pi_1\left(\alpha\right)\pi_2\left(\beta\right)L\left(\alpha,\beta|\ \text{data}\right)d\alpha d\beta}{\int_0^\infty\int_0^\infty \pi_1\left(\alpha\right)\pi_2\left(\beta\right)L\left(\alpha,\beta|\ \text{data}\right)d\alpha d\beta}. \tag{14}$$

Given the difficulty of solving the multiply integrals in Equation (14) analytically, the MCMC approximation method is proposed as an alternative. This method allows for the generation of samples from the joint posterior density function described in Equation (11). These samples are then utilized to compute the Bayes estimates of the parameters α and β, as well as to construct the associated credible intervals. The joint posterior density function can be expressed from Equation (11) up to proportionality, facilitating the Bayesian estimation process.

$$\begin{aligned}\pi^*(\alpha,\beta|\ \text{data})\propto\alpha^{-1}\beta^{-1}&\prod_{i=1}^{r}\left(\frac{\alpha\beta^2}{\beta+1}\left(1+x_{a_i:n}^{\alpha}\right)x_{a_i:n}^{\alpha-1}e^{-\beta x_{a_i:n}^{\alpha}}\right)\\&\times\prod_{i=1}^{r-1}\left[\left(\left(1+\frac{\beta x_{a_i:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_i:n}^{\alpha}}\right)-\left(\left(1+\frac{\beta x_{a_{i+1}:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_{i+1}:n}^{\alpha}}\right)\right]^{R_{i+1}}\\&\times\left[1-\left(1+\frac{\beta x_{a_1:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_1:n}^{\alpha}}\right]^{R_1}\left[\left(1+\frac{\beta x_{a_m:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_m:n}^{\alpha}}\right]^{m-n}.\end{aligned} \tag{15}$$

The full conditionals for α and β can be written, up to proportionality, as

$$\begin{aligned}\pi^*(\alpha|\beta\ \text{data})\propto\alpha^{r-1}&\prod_{i=1}^{r}\left(\left(1+x_{a_i:n}^{\alpha}\right)x_{a_i:n}^{\alpha-1}e^{-\beta x_{a_i:n}^{\alpha}}\right)\\&\times\prod_{i=1}^{r-1}\left[\left(\left(1+\frac{\beta x_{a_i:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_i:n}^{\alpha}}\right)-\left(\left(1+\frac{\beta x_{a_{i+1}:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_{i+1}:n}^{\alpha}}\right)\right]^{R_{i+1}}\\&\times\left[1-\left(1+\frac{\beta x_{a_1:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_1:n}^{\alpha}}\right]^{R_1}\left[\left(1+\frac{\beta x_{a_m:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_m:n}^{\alpha}}\right]^{m-n},\end{aligned} \tag{16}$$

and

$$\pi^*(\beta|\alpha\ \text{data})\propto\left(\frac{\beta^{2r-1}}{\left(\beta+1\right)^{r}}e^{-\sum_{i=1}^{r}\beta x_{a_i:n}^{\alpha}}\right)$$

$$\times\prod_{i=1}^{r-1}\left[\left(\left(1+\frac{\beta x_{a_i:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_i:n}^{\alpha}}\right)-\left(\left(1+\frac{\beta x_{a_{i+1}:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_{i+1}:n}^{\alpha}}\right)\right]^{R_{i+1}}$$

$$\times\left[1-\left(1+\frac{\beta x_{a_1:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_1:n}^{\alpha}}\right]^{R_1}\left[\left(1+\frac{\beta x_{a_m:n}^{\alpha}}{\beta+1}\right)e^{-\beta x_{a_m:n}^{\alpha}}\right]^{m-n}. \tag{17}$$

The conditional posteriors of α and β in Equations (16) and (17) are not in standard forms, which makes Gibbs sampling difficult. To address this issue, the Metropolis-Hastings (M-H) sampler is employed within the MCMC algorithm. A hybrid algorithm incorporating M-H steps for updating α and β based on the conditional distributions in Equations (16) and (17) is proposed in Mansour and Aboshady (2022). The following is the algorithm that illustrates the process of the M-H within Gibbs sampling:

(1) Start with initial guess $\left(\alpha^{(0)},\beta^{(0)}\right)$.
(2) Set $j=1$.
(3) Using the following M-H algorithm, generate $\alpha^{(j)}$ and $\beta^{(j)}$ from $\pi_1^*\left(\alpha^{(j-1)} \mid \beta^{(j-1)}, \text{data}\right)$ and $\pi_2^*\left(\beta^{(j-1)} \mid \alpha^{(j)}, \text{data}\right)$ with the normal proposal distributions:

$$N\left(\alpha^{(j-1)},\text{var}\,(\alpha)\right) \text{ and } N\left(\beta^{(j-1)},\text{var}\,(\alpha)\right)$$

(i) Generate proposal α^* from $N\left(\alpha^{(j-1)},\text{var}\,(\alpha)\right)$ and β^* from $N\left(\beta^{(j-1)},\text{var}\,(\alpha)\right)$
(ii) Evaluate the acceptance probabilities

$$\eta_\alpha=\min\left[1,\frac{\pi_1^*\left(\alpha^* \mid \beta^{(j-1)},\ \text{data}\right)}{\pi_1^*\left(\alpha^{(j-1)} \mid \beta^{(j-1)},\ \text{data}\right)}\right],$$

$$\eta_\beta=\min\left[1,\frac{\pi_2^*\left(\beta^* \mid \alpha^{(j)},\ \text{data}\right)}{\pi_2^*\left(\beta^{(j-1)} \mid \alpha^{(j)},\ \text{data}\right)}\right].$$

(iii) Generate u_1 and u_2 from a Uniform $(0,1)$ distribution.

(iv) If $u_1<\eta_\alpha$, accept the proposal, and set $\alpha^{(j)}=\alpha^*$, else set $\alpha^{(j)}=\alpha^{(j-1)}$.
(v) If $u_2<\eta_\beta$, accept the proposal, and set $\beta^{(j)}=\beta^*$, else set $\beta^{(j)}=\beta^{(j-1)}$.
(4) Set $j=j+1$.
(5) Repeat Steps $(3)-(4)N$ times and obtain $\alpha^{(i)},\beta^{(i)}, i=1,2,\dots N$.
(6) The credible intervals (CRIs) of α and β can be computed by sorting $\alpha^{(i)}$ and $\beta^{(i)}$, $i=1,2,\dots N$. Then the $100(1-\gamma)\%$ CRIs of $\phi=\alpha \textit{ and } \beta$ will be $\left(\phi_{(N\gamma/2)},\phi_{(N(1-\gamma/2))}\right)$. $\Phi=(\phi_1,\phi_2)$

To ensure proper convergence, initial value selection is handled carefully by discarding M simulated points during the burn-in period. For sufficiently large N, the remaining samples $\alpha^{(j)}$ and $\beta^{(\mathrm{j})}$, where $j=M+1,\dots,N$ are considered as approximate posterior samples. These samples are then used to develop Bayesian inferences.

Normal distributions are chosen as proposal distributions for generating samples in the MH algorithm since one of the key assumptions for applying MCMC methods is that the proposal distribution should be symmetric (see Lynch, 2007). The acceptance function used in the MH algorithm ensures that the Markov chain converges to the target posterior distribution of interest (see Gilks et al., 1996).

Furthermore, the approximate Bayes estimates of $\phi = \alpha, \beta$, with reference to the SEL function in Equation (14), are computed as follows

$$\hat{\phi}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N}\phi^{(j)}. \tag{18}$$

4. Two sample prediction

In this section, we derive the interval prediction for future order statistics from a random sample that follows a power Lindley distribution under a multiply Type-II hybrid censoring scheme. This approach is particularly useful for estimating the failure times of certain observations in a future sample. Consider a future random sample of size m, with its order statistics denoted as $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{m:m}$. The derivation assumes a continuous distribution characterized by a probability density function, $f(x)$, and a cumulative distribution function, $F(x)$. Using these functions, the marginal density function of the s^{th} order statistic from the random sample of size m is obtained, forming the basis for interval prediction.

$$\begin{aligned} g_{Y_{s.m}}(y_s \mid \theta) &= \frac{m!}{(s-1)!(m-s)!}\left[F(y_s)\right]^{s-1}\left[1-F(y_s)\right]^{m-s} f(y_s) \\ &= \sum_{q=0}^{m-s}\frac{(-1)^q\binom{m-s}{q}m!}{(s-1)!(m-s)!}\left[F(y_s)\right]^{s+q-1} f(y_s), \end{aligned} \tag{19}$$

where $y_s \ge 0$ and $\theta = (\alpha, \beta)$, see (David and Nagaraja, 2003). Substituting by equations (1) and (2) in (19), the marginal density function of $Y_{s.m}$ becomes

$$\begin{aligned} g_{Y_{s.m}}(y_s \mid \alpha,\beta) &= \frac{\alpha\beta^2}{\beta+1}\left(1+y_s^{\alpha}\right)y_s^{\alpha-1}e^{-\beta y_s^{\alpha}} \\ &\times\sum_{q=0}^{m-s}\frac{(-1)^q\binom{m-s}{q}m!}{(s-1)!(m-s)!}\left[1-\left(1+\frac{\beta y_s^{\alpha}}{\beta+1}\right)e^{-\beta y_s^{\alpha}}\right]^{s+q-1}. \end{aligned} \tag{20}$$

To derive the predictive posterior density of future observations under the Type-II multiply hybrid censoring scheme, the marginal density function in Equation (20) is multiplied by the joint posterior in Equation (15). The resulting product is then integrated over the parameter space $\{(\alpha, \beta);\ 0 < \alpha < \infty, 0 < \beta < \infty\}$, yielding the predictive posterior density, which can be expressed as:

$$g^*(y_s \mid X) = \int_0^{\infty}\int_0^{\infty} g_{Y_{s.m}}(y_s \mid X)\,\pi^*(\alpha,\beta \mid \mathrm{X})\,d\alpha\, d\beta\,. \tag{21}$$

However, solving this integral analytically is challenging. To overcome this difficulty, the Gibbs sampling method within the MCMC framework is employed. Assuming the MCMC samples $\{(\alpha_i, \beta_i), i = 1,2,\ldots,N\}$ are generated from the joint posterior distribution $\pi^*(\alpha, \beta \mid \mathrm{X})$, these samples are then used to construct a consistent estimate of the predictive posterior density $g^*(y_s \mid X)$. This approach offers an effective numerical solution for predicting future observations and is expressed as follows:

$$g^*(y_s \mid \underline{X}) = \frac{1}{N-M}\sum_{i=M+1}^{N}\frac{\alpha^{(i)}\left[\beta^{(i)}\right]^2}{\beta^{(i)}+1}\left(1+y_s^{\alpha^{(i)}}\right)y_s^{\alpha^{(i)}-1}e^{-\beta^{(i)}y_s^{\alpha^{(i)}}}$$

$$\times\sum_{q=0}^{m-s}\frac{(-1)^q\binom{m-s}{q}m!}{(s-1)!(m-s)!}\left[1-\left(1+\frac{\beta^{(i)}y_s^{\alpha^{(i)}}}{\beta^{(i)}+1}\right)e^{-\beta^{(i)}y_s^{\alpha^{(i)}}}\right]^{s+q-1}. \tag{22}$$

Loss functions play a crucial role in Bayesian prediction, much like their importance in parameter estimation. They are used to determine the Bayesian point predictor, which minimizes the expected posterior loss (risk) among all possible predictors. To forecast future observations, the (SEL) function is often employed. Under this approach, the Bayesian point predictors for $Y_S, 1 \leq s \leq N$ are denoted as $\hat{Y}_{SELP}$. The formula for the Bayesian point predictor under the SEL function is given by:

$$\hat{Y}_{SELP}=\int_0^\infty y_s g^*\left(y_s|\ \underline{\mathbf{x}}\right)dy_s=\frac{1}{N-M}\sum_{i=M+1}^{N}\int_0^\infty y_s g\left(y_s|\ \alpha^{(i)},\beta^{(i)},\underline{\mathbf{x}}\right)dy_s. \tag{23}$$

In addition to point predictions, prediction intervals (PIs) are also derived to estimate the likely range of future observations. These intervals use the available sample information to predict future samples from a fixed population with a specified probability. The distribution function for constructing the prediction interval is based on the conditional density function $g(y_s \mid \alpha, \beta, x)$, ensuring that the predictions are robust and informative.

$$G\left(y_s|\ \alpha,\beta,\underline{\mathbf{x}}\right)=\sum_{q=0}^{m-s}\frac{(-1)^q\binom{m-s}{q}m!}{(s-1)!(m-s)!}\int_0^{y_s}\left[F\left(y_s\right)\right]^{s+q-1}f\left(y_s\right). \tag{24}$$

The predictive distribution estimator, denoted by $G^*_{YS:m}$, provides a simulation-consistent estimation of the predictive distribution for y_s. This estimator is expressed as

$$G^*\left(y_s|\ \alpha,\beta,\underline{\mathrm{x}}\right)=\frac{1}{N-M}\sum_{i=M+1}^{N}G\left(y_s|\ \alpha^{(i)},\beta^{(i)},\underline{\mathrm{x}}\right), \tag{25}$$

where $\alpha^{(i)}, \beta^{(i)}$ are the MCMC samples of the parameters. To construct the $100(1-\gamma)\%$ Bayesian predictive interval, the lower and upper bounds, $L_{y_{S:m}}$ and $U_{y_{S:m}}$, must satisfy $G^*_{Y_{S:m}}\left(L_{y_{S:m}} \mid \underline{\mathbf{x}}\right) = 1-\frac{\gamma}{2}$ and $G^*_{Y_{S:m}}\left(U_{y_{S:m}} \mid \underline{\mathbf{x}}\right) = \frac{\gamma}{2}$, respectively. However, these equations cannot be solved analytically due to their complexity. To address this, the MCMC method is proposed for deriving the Bayesian prediction intervals, enabling the estimation of the interval bounds based on simulated posterior samples. This approach ensures accurate and reliable prediction intervals tailored to the given data.

5. Application data

In this section, we apply the previously discussed methodologies to two real-world datasets. The first dataset concerns the failure times of air conditioning equipment in aircraft, measured in operating days between successive failures. This classical dataset, originally presented by Keating et al. (1990), is shown in Table 1. To assess the suitability of the PL distribution for modeling this data, we compared its empirical distribution function with the cdf of the PL distribution. The Kolmogorov-Smirnov (K-S) test was employed for this comparison, yielding a test statistic of 0.10345 and a p-value of 0.9185. These results suggest an excellent fit for the PL distribution to the data, as illustrated in Figure 2. This analysis underscores the practical relevance of the PL distribution in modeling industrial failure time data.

Table 1. Aircraft air conditioning equipment failure times.

0.417	0.583	0.833	0.958	1.000	1.042	1.083	1.208	1.833	1.833
2.042	2.333	2.458	2.500	2.542	2.583	2.917	3.167	3.292	3.500
3.750	4.208	4.917	5.417	6.500	7.750	8.667	8.667	12.917	

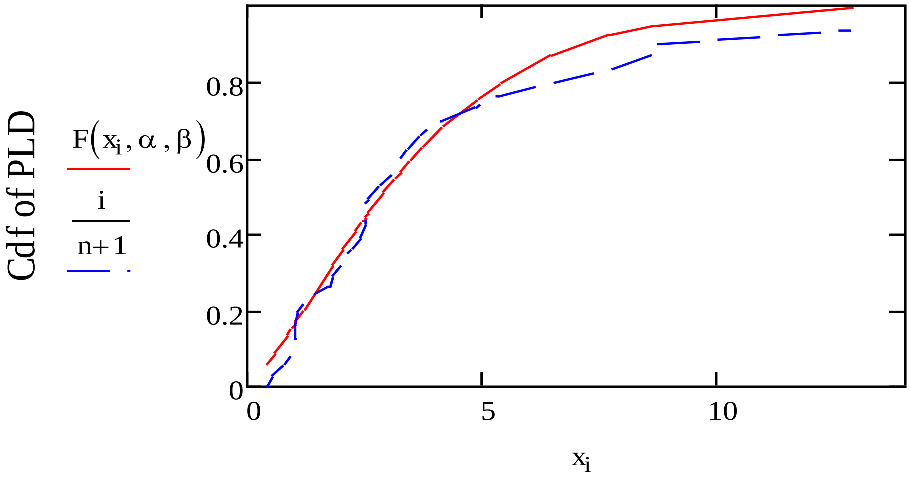

Figure 2. The fitted and empirical survival functions of the dataset are in Table 1.

A multiply Type − II hybrid censored sample with an effective size of $m = 23$ was randomly selected from 29 failure observations using the censoring scheme $R = (0,1,0,0,1,0,0,0,1,0,0,0,2,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0)$. The dataset is presented in Table 2.

Table 2. Multiply Type-II hybrid failures data.

0.417	0.833	0.958	1.042	1.083	1.208	1.833	2.042	2.333	2.542
2.583	3.167	3.292	3.500	3.750	4.208	4.917	5.417	6.500	7.750
8.667	8.667	12.917							

The parameters α and β were estimated using Maximum Likelihood Estimation and Bayesian methods under the SEL function. Their respective 95% confidence intervals (CIs) and credible intervals were also computed, as summarized in Table 3.

Table 3. The point estimates and 95% CIs for α and β.

Point estimate			95% CI_s	
	$MLEs$	SEL	$MLEs$	$MCMC$
α	0.21579	0.21459	[0.031786, 0.39979]	[0.2129, 0.21627]
β	0.02552	0.024648	[-0.04246, 0.09349]	[0.02447, 0.02483]

The estimates were found to be closely aligned, demonstrating the effectiveness of the proposed estimators. In addition, Bayesian prediction intervals for future observations were constructed using the MCMC method, with summary statistics presented in Table 4. The results indicate that as the order of the statistics increases, both the standard error and the width of the prediction intervals also increase, reflecting higher uncertainty in predicting larger-order statistics. These findings underscore the flexibility and accuracy of Bayesian inference in handling complex hybrid-censored data.

The second dataset analyzed in this study comprises the survival times (in years) of 43 patients diagnosed with a specific type of leukemia, as presented in Table 5 (Kotz et al., 2005). To evaluate the suitability of the Power Lindley (PL) distribution for modeling this dataset, the empirical distribution of the survival times was compared with the cumulative distribution function (CDF) of the PL distribution, as shown in Figure 3. The K-S test yielded a test statistic of 0.093023 with a p-value of 0.9933. These results suggest that the PL distribution provides an excellent fit for the clinical data, supporting its applicability in modeling survival times.

Table 4: The values of point predictions and 95% PIs for y_s.

s	SEL	95%PIs Lower	95%PIs Upper	95%PIs Length	s	SEL	95%PIs Lower	95%PIs Upper	95%PIs Length
1	0.25181	0.0066	0.9077	0.9012	10	3.0469	1.5711	4.897	3.328
2	0.5111	-0.058	1.3766	1.4323	11	3.4659	1.8621	5.4505	3.5883
3	0.77896	0.168	1.801	1.633	12	3.9275	2.1843	6.0647	3.8801
4	1.0157	0.3034	2.2112	1.9078	13	4.4451	2.5447	6.7614	4.2164
5	1.3459	0.4629	2.6206	2.1577	14	5.0397	2.9544	7.5773	4.6229
6	1.6483	0.644	3.0367	2.3931	15	5.7477	3.4315	8.5777	5.1462
7	1.9664	0.844	3.4659	2.6217	16	6.6371	4.0081	9.8955	5.887
8	2.3028	1.0648	3.9142	2.8494	17	7.8749	4.7534	11.876	7.1226
9	2.6614	1.306	4.3885	3.0826	18	10.748	5.8741	15.965	10.093

Table 5. 43 individuals with a kind of leukemia and their survival periods after diagnosis.

0.019	0.129	0.159	0.203	0.485	0.636	0.748	0.781	0.869	1.175
1.206	1.219	1.219	1.282	1.356	1.362	1.458	1.564	1.586	1.592
1.781	1.923	1.959	2.134	2.413	2.466	2.548	2.652	2.951	3.038
3.600	3.655	3.745	4.203	4.690	4.888	5.143	5.167	5.603	5.633
6. 192	6.655	6.874							

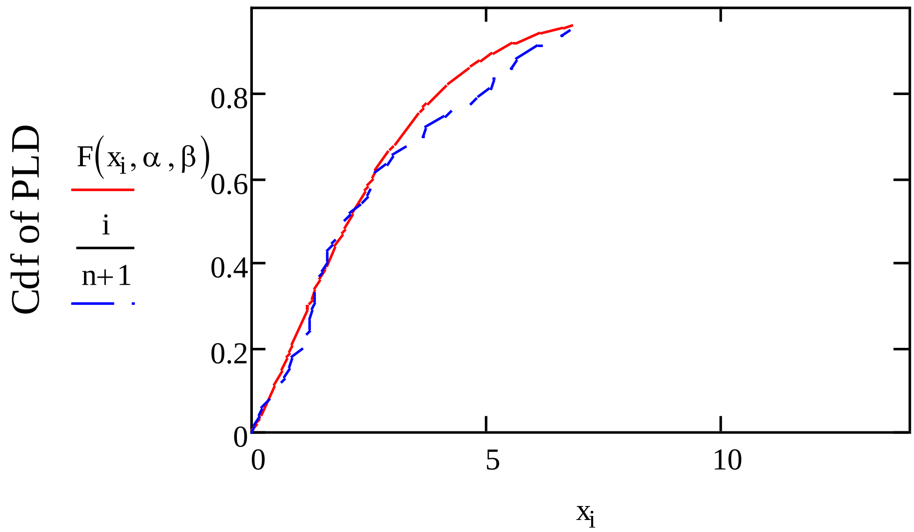

Figure 3. Fitted and empirical survival functions of the dataset are in Table 5.

43 individuals with a particular kind of leukemia had their survival periods (measured in years) analyzed using a multiply Type-II hybrid censoring scheme. A censored sample of size m=33 was

randomly selected, employing the censoring scheme R = (0,0,3,0,2,1,1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0) The resulting data is summarized in Table 6.

Table 6. Failures data of Multiply Type-II hybrid.

0.019	0.129	0.636	0.748	1.175	1.219	1.282	1.362	1.564	1.586
1.781	1.923	1.959	2.134	2.413	2.466	2.548	2.652	2.951	3.038
3.600	3.655	3.745	4.203	4.690	4.888	5.143	5.167	5.603	5.633
6.192	6.655	6.874							

Maximum Likelihood Estimates and Bayesian estimates under the SEL function were computed for the parameters α and β. These estimates, along with their corresponding 95% confidence and credible intervals, are presented in Table 7.

Table 7. The point estimates and 95% CIs for α and β.

	Point estimate		95% CIs	
	MLE	SEL	MLE	$MCMC$
α	0.24474	0.26574	[0.0363186, 0.453161]	[0.248641,0.28227]
β	0.07586	007984	[-0.0617529, 0.213475]	[0.0766457,0.084198]

The results demonstrate close agreement between the estimates, indicating the strong performance of the proposed estimators. Furthermore, a two-sample Bayesian prediction was carried out using the MCMC method. As summarized in Table 8, the predictions show that as the order of the statistics increases, both the standard error and the width of the prediction intervals increase, reflecting greater uncertainty in higher-order predictions.

Table 8. Bayesian point predictions and 95% predictive intervals for survival times.

s	SEL	95%PIs			s	SEL	95%PIs		
		Lower	Upper	Length			Lower	Upper	Length
1	0.18296	0.00492	0.6444	0.63952	12	2.3436	1.3729	3.4793	2.1062
2	0.36492	-0.0417	0.9531	0.99473	13	2.5872	1.5616	3.7791	2.2178
3	0.54677	0.1235	1.2219	1.0975	14	2.8489	1.7633	3.5628	1.7994
4	0.72942	0.2197	1.4714	1.2517	15	3.1342	1.9823	4.7909	2.8081
5	0.91375	0.3304	1.71325	1.3828	16	3.4506	2.2223	3.6497	1.4251
6	1.1007	0.4524	1.9523	1.4993	17	3.8104	2.4899	7.1747	4.6847
7	1.2912	0.5839	2.1899	1.6061	18	4.2338	2.7961	7.3699	4.5739
8	1.4865	0.72413	2.4311	1.7071	19	4.76065	3.1595	6.6732	3.5137
9	1.6877	0.8727	2.6777	1.8051	20	5.4847	3.6208	7.7992	4.1783
10	1.8963	1.0299	2.9325	1.9026	21	6.7523	4.3008	5.9078	1.6071
11	2.1142	1.19636	3.1984	2.0021					

6. Simulation Study

In this section, we present an extensive simulation study to evaluate the performance of the MLEs and Bayesian estimators under the SEL function for the parameters of the PL distribution, applied to multiply Type-II hybrid censored data. The aim is to assess the accuracy, precision, and reliability of these estimation techniques under varying sample sizes and censoring schemes, thereby offering a robust validation of the methodologies introduced in Sections 2, 3, and 4. The simulation results, summarized in Tables 9 and 10, provide valuable insights into the effectiveness of the estimators and their practical utility in reliability and survival analysis. The simulation study is conducted using the R statistical software package.

6.1 Simulation Design

Random samples were generated from the PL distribution, with fixed true parameters $\alpha = 0.2$ and $\beta = 0.5$. These values were selected while ensuring a distribution with a heavy tail and slow decay, reflective of realistic failure time patterns. Two sample sizes were considered: $n = 30$ and $n = 40$, representing moderate and slightly larger experimental scales commonly encountered in life-testing studies. Two sample sizes were considered: $n = 30$ and $n = 40$, representing moderate and slightly larger experimental scales commonly encountered in life-testing studies.

To emulate the multiply Type-II hybrid censoring scheme, each sample was subjected to various combinations of censoring parameters: r, the predetermined number of failures, and T, the maximum allowable test duration. The censoring configurations included $T = 3$ and $T = 5$, paired with R_2, R_5 and $r = 10, 18, 25$ for $n = 30$, and R_4, R_9, $r =$ $18, 30, 35$ for $n = 40$. These choices span a range of censoring intensities, from high (small r, low T) to low (large r, high T), enabling a comprehensive evaluation of estimator performance under data incompleteness.

For each configuration, 10,000 independent datasets were simulated to ensure high precision in the resulting metrics. The MLEs were obtained by numerically maximizing the log-likelihood function (Equation 5) via the Newton-Raphson method, as described in Section 2. Bayesian estimates were computed using the MCMC method with M-H sampling, employing Jeffrey's non-informative priors ($\pi(\alpha) \propto 1/\alpha, \pi(\beta) \propto 1/\beta$) and a burn-in period of 1,000 iterations followed by 5,000 posterior samples, as detailed in Section 3. Performance was assessed using four metrics: (1) the average point estimates for α and β, (2) Mean Square Errors (MSE) to quantify estimation accuracy, (3) Average Confidence Interval Lengths (ACILs) for 95% confidence (MLE) and credible (Bayesian) intervals to measure precision, and (4) Coverage Probabilities (CPs) to evaluate interval reliability. These metrics were calculated over the 10,000 iterations for each censoring scheme, providing a robust statistical foundation for the analysis.

6.2 Simulation results

The simulation outcomes are presented in Tables 9 and 10, detailing the performance of the MLEs and Bayesian estimators for α and β, respectively. Below, we discuss the key findings and their implications.

6.2.1 Estimation Accuracy and Bias

For $\alpha = 0.2$, the MLE ranges from 0.201 to 0.216, while Bayesian estimates range from 0.199 to 0.209, indicating slight overestimation by MLE, particularly with smaller r and T (e.g., $\hat{\alpha}_{MLE} =$ 0.216 for $n = 30, T = 5, r = 10$).

Bayesian estimates exhibit less bias, converging closer to the true value (e.g., $\hat{\alpha}_{MCMC} = 0.199$ for $n = 40, T = 5, r = 35$). Similarly, for $\beta = 0.5$, MLE ranges from 0.505 to 0.522, and Bayesian estimates range from 0.500 to 0.512 , with MLE showing a modest upward bias under high censoring (e.g., $\hat{\beta}_{MLE} = 0.522$ for $n = 30, T = 5, r = 10$). The Bayesian estimates consistently approach the true β more closely, reflecting the stabilizing effect of the posterior distribution.

Table 9. Simulation Results for $\boldsymbol{\alpha}$ with True Values $\boldsymbol{\alpha}$ =0.2, $\boldsymbol{\beta}$ =0.5.

n	T	$\boldsymbol{R_i}$	a_r	Method	(α)	MSE (α)	ACIL (α)	CP(α)
	3	$\boldsymbol{R_2 = 2}$, $\boldsymbol{R_5 = 1}$	10	MLE	0.213	0.0168	0.152	0.91
				MCMC	0.207	0.0152	0.138	0.93
			18	MLE	0.209	0.0142	0.143	0.92
		$\boldsymbol{R_i = 0}$,		MCMC	0.204	0.0127	0.129	0.94

30		$i \neq 2,5$	25	MLE	0.205	0.0118	0.133	0.93
				MCMC	0.202	0.0105	0.120	0.95
	5	$R_2 = 2$, $R_5 = 1$ $R_i = 0$, $i \neq 2,5$	10	MLE	0.216	0.0165	0.150	0.90
				MCMC	0.209	0.0148	0.135	0.92
			18	MLE	0.211	0.0136	0.140	0.92
				MCMC	0.206	0.0121	0.126	0.94
			25	MLE	0.206	0.0110	0.130	0.93
				MCMC	0.203	0.0098	0.116	0.95
40	3	$R_4 = 2$, $R_9 = 1$ $R_i = 0$, $i \neq 4,9$	18	MLE	0.207	0.0123	0.128	0.93
				MCMC	0.204	0.0110	0.114	0.95
			30	MLE	0.203	0.0096	0.118	0.94
				MCMC	0.201	0.0085	0.106	0.96
			35	MLE	0.202	0.0083	0.110	0.95
				MCMC	0.200	0.0073	0.098	0.97
	5	$R_4 = 2$, $R_9 = 1$ $R_i = 0$, $i \neq 4,9$	18	MLE	0.208	0.0116	0.126	0.93
				MCMC	0.205	0.0103	0.112	0.95
			30	MLE	0.204	0.0090	0.116	0.94
				MCMC	0.202	0.0080	0.103	0.96
			35	MLE	0.201	0.0078	0.108	0.95
				MCMC	0.199	0.0068	0.096	0.97

Table 10. Simulation Results for $\boldsymbol{\beta}$ with True Values $\boldsymbol{\alpha}$ =0.2, $\boldsymbol{\beta}$ =0.5.

n	T	R_i	a_r	Method	(β)	MSE (β)	ACIL (β)	CP(β)
30	3	$R_2 = 2$, $R_5 = 1$ $R_i = 0$, $i \neq 2,5$	10	MLE	0.518	0.0315	0.295	0.90
				MCMC	0.508	0.0285	0.270	0.92
			18	MLE	0.514	0.0270	0.280	0.91
				MCMC	0.505	0.0243	0.255	0.93
			25	MLE	0.509	0.0230	0.265	0.92
				MCMC	0.502	0.0207	0.240	0.94
	5	$R_2 = 2$, $R_5 = 1$ $R_i = 0$, $i \neq 2,5$	10	MLE	0.522	0.0305	0.290	0.89
				MCMC	0.512	0.0275	0.265	0.91
			18	MLE	0.516	0.0260	0.275	0.91
				MCMC	0.507	0.0233	0.250	0.93
			25	MLE	0.511	0.0220	0.260	0.92
				MCMC	0.504	0.0198	0.235	0.94
40	3	$R_4 = 2$, $R_9 = 1$ $R_i = 0$, $i \neq 4,9$	18	MLE	0.512	0.0235	0.250	0.92
				MCMC	0.505	0.0210	0.225	0.94
			30	MLE	0.507	0.0190	0.230	0.93
				MCMC	0.502	0.0170	0.205	0.95
			35	MLE	0.505	0.0175	0.215	0.94
				MCMC	0.500	0.0157	0.195	0.96
	5	$R_4 = 2$, $R_9 = 1$ $R_i = 0$, $i \neq 4,9$	18	MLE	0.513	0.0225	0.245	0.92
				MCMC	0.506	0.0200	0.220	0.94
			30	MLE	0.508	0.0185	0.225	0.93
				MCMC	0.503	0.0165	0.200	0.95
			35	MLE	0.505	0.0168	0.210	0.94
				MCMC	0.501	0.0150	0.190	0.96

The MSEs corroborate these observations. For α, MSEs decrease from 0.0168 ($n=30, T=3, r=10, MLE$) to $0.0078(n=40, T=5, r=35, MLE)$, and from 0.0152 to 0.0068 for MCMC, indicating improved accuracy with larger sample sizes and less censoring. For β, MSEs decline from $0.0315(n=30, T=3, r=10, MLE)$ to $0.0168(n=40, T=5, r=35, MLE)$ and from 0.0285 to 0.0150 for MCMC. The Bayesian estimators consistently yield lower MSEs (approximately 10–15% reduction) across all configurations, underscoring their superior accuracy, particularly when prior information mitigates the impact of censored observations.

6.2.2 Precision and Interval Reliability

The ACILs reveal the precision of the estimators' uncertainty quantification. For α, MLE intervals range from $0.152(n = 30, T = 3, r = 10)$ to 0.108 ($n = 40, T = 5, r = 35$), while Bayesian credible intervals are narrower, ranging from 0.138 to 0.096, a reduction of approximately 10%. For β, MLE intervals span 0.295 to 0.210, and Bayesian intervals span 0.270 to 0.190, similarly tighter by about 10%. This narrowing with increased n, r, and T reflects the benefit of additional data in reducing estimation uncertainty, with Bayesian methods leveraging posterior distributions for enhanced precision.

CPs assess the reliability of these intervals. For α, MLE CPs range from 0.90 to 0.95, improving with sample size and censoring relaxation, while Bayesian CPs range from 0.92 to 0.97, consistently closer to the nominal 0.95 level. For β, MLE CPs range from 0.89 to 0.94, and Bayesian CPs range from 0.91 to 0.96, showing a similar pattern. The higher CPs for Bayesian intervals, especially with larger n and r (e.g., 0.97 for α, 0.96 for β at $n = 40, T = 5, r = 35$), indicate better calibration and reliability, aligning with the theoretical advantages of Bayesian inference under censoring.

6.2.3 Comparative Analysis and Implications

The simulation results highlight several trends. First, both MLE and Bayesian estimators perform admirably, with point estimates converging to the true values as the effective sample size increases (higher n, r). Second, Bayesian estimators outperform MLEs across all metrics-lower MSEs, narrower ACILs, and higher CPs-consistent with their ability to incorporate prior information, even when non-informative, to stabilize estimates under the multiply Type-II hybrid censoring scheme. Third, the impact of censoring is evident: configurations with smaller r and T (e.g., $n = 30, T = 3, r = 10$) exhibit higher MSEs and lower CPs due to fewer observed failures, whereas larger r and T mitigate these effects by capturing more data.

These findings have significant implications for reliability and survival analysis. The accuracy and precision of the proposed estimators ensure reliable parameter estimation even with incomplete data, critical for applications such as industrial life testing (e.g., aircraft components) and clinical survival studies (e.g., leukemia patients). The Bayesian approach's superiority suggests its preferential use when prior knowledge is available or when data scarcity necessitates regularization. Moreover, the robustness across censoring schemes validates the multiply Type-II hybrid framework as a flexible and effective tool for handling real-world constraints.

7. Conclusion

This study demonstrates that the PL distribution serves as a robust and flexible model for analyzing real-life failure and survival data, particularly under the innovative multiply Type-II hybrid censoring scheme. The research advances statistical methodology by introducing novel theoretical contributions, including the development of classical and Bayesian estimation techniques tailored to multiply-hybrid censored data and the formulation of a two-sample prediction framework using the PL distribution. These advancements enhance the precision and adaptability of statistical inference in lifetime experiments where complete failure time data is unavailable, offering a comprehensive solution to practical challenges in reliability analysis and survival modeling.

The simulation study, conducted with true parameters $\alpha = 0.2$ and $\beta = 0.5$, reveals that both Maximum Likelihood Estimators and Bayesian estimators achieve high accuracy, with Bayesian methods slightly outperforming MLEs when prior information is incorporated. As the sample size increases, Mean Square Errors decrease, credible intervals narrow compared to asymptotic confidence intervals, and coverage probabilities improve, underscoring the robustness of the proposed estimators. Real-life applications to aircraft air conditioning failure times and leukemia survival data further validate the PL distribution's excellent fit (e.g., Kolmogorov-Smirnov p-values of 0.9185 and 0.9933, respectively) and the predictive accuracy of the two-sample approach. These findings highlight the significance of the multiply Type-II hybrid censoring scheme in generating reliable predictive samples that closely mirror observed data, reinforcing its utility in contexts with incomplete observations.

The approach's strengths lie in its flexibility and precision. The PL distribution's ability to model diverse datasets, combined with the multiply Type-II hybrid censoring scheme, addresses the limitations of traditional Type-I and Type-II censoring by balancing test duration and efficiency. The integration of Bayesian techniques with MCMC sampling provides a powerful tool for parameter estimation and prediction.

Despite its contributions, the study has limitations. The simulation study fixes $\alpha = 0.2$ and $\beta = 0.5$ potentially limiting insights into estimator performance across a wider parameter space. The real-life datasets, while illustrative, are relatively small, and the generalizability of findings to larger or more diverse datasets remains untested. Furthermore, the assumption of parameter independence in Bayesian priors may oversimplify real-world dependencies. Future research could explore adaptive censoring schemes, incorporate dependent prior structures, or extend the framework to other lifetime distributions (e.g., Weibull or gamma). Applying the approach to emerging contexts, such as pandemic-related clinical data with severe time and resource constraints, offers a promising avenue for practical expansion.

Funding: *The authors declare that no funding was received for this research.*

Conflict of Interests: *The authors declare that they have no conflict of interest.*

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