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Comparative Study of Estimation Methods for Kumaraswamy Weibull Regression Model: An Application to Economic Value-Added Data

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Abstract

This study investigates several parameter estimation techniques for the Kumaraswamy Weibull regression model using economic value-added data. Five methods are compared: maximum likelihood estimation (MLE), ordinary least squares (OLS), weighted least squares (WLS), Cramér-von Mises (CVM), and Anderson-Darling (AD). The analysis, based on quarterly data from five firms over 24 periods, shows that MLE consistently achieves the lowest values across the three information criteria (AIC, BIC, and HQIC). After identifying MLE as the optimal estimation technique, parameters for both Kumaraswamy Weibull and standard Weibull models were estimated. Statistical tests reveal the superiority of the Kumaraswamy Weibull model in handling economic value-added data, yielding a higher p-value (0.196) compared to the standard Weibull model (0.185). Additionally, the regression model based on the Kumaraswamy Weibull distribution demonstrates a superior fit ($R^2 = 0.96$ compared to 0.84) and aligns more closely with economic theory. In particular, economic value-added is found to be positively related to firm size and negatively related to both leverage and collection periods. The findings offer important methodological insights for selecting appropriate distributions and estimation methods in complex financial data modeling.

Mathematical Subject Classification: 62F10, 62J05

1. Introduction

Regression models are a fundamental tool for understanding relationships between variables across various disciplines. Recently, researchers have shown increased attention to the development of new and more precise regression models, especially in finance and economics, where classical models sometimes fall short. Although the standard Weibull distribution is common in reliability and survival analysis, its limitations become apparent in financial time series, which often display bathtub-shaped or unimodal hazard functions (Lai et al., 2003; Zhang & Xie, 2007).

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The Kumaraswamy Weibull distribution, introduced by Cordeiro et al. (2010), offers enhanced flexibility through its four parameters (two shapes, one scale, and one location). This flexibility is particularly useful for modeling financial metrics like economic value added, which typically exhibit non-normality, skewness, and heavy tails (Bali et al., 2008). Unlike the standard Weibull—which restricts the failure rate to a monotonic increase or decrease—the Kumaraswamy Weibull can accommodate bathtub, unimodal, and other complex hazard rate shapes (Al-Mofleh, 2016). This blend of desirable mathematical properties makes it an attractive choice for rigorous statistical inference (Cordeiro et al., 2013).

Kumaraswamy-type distributions have proven effective in various fields such as reliability and survival analysis. For example, the Kumaraswamy generalized Rayleigh distribution (Gomes et al., 2014) and the Kumaraswamy transmuted-G family (Afify et al., 2018) demonstrate superior flexibility over traditional distributions, indicating their potential utility in financial data analysis.

Earlier studies, such as Ortega et al. (2003) on censored observations in generalized log-gamma models and Hashimoto et al. (2010) on interval-censored data in log exponentiated Weibull models, laid the groundwork for improved parameter estimation techniques. Building on this foundation, Cordeiro et al. (2011) proposed new estimation procedures for regression models based on the exponentiated generalized gamma distribution. Our work revisits these classical estimation challenges in the context of more complex distributions (Choi & Bulgren, 1968; MacDonald, 1971).

Recent comparative studies have increasingly focused on estimation methods. Ergenç and Şenoğlu (2023) demonstrated via Monte Carlo simulations that MLE outperforms alternative methods for the Kumaraswamy Weibull distribution, with AD estimation ranking second. In another study, Ali et al. (2020) compared ten estimation methods for the flexible Weibull distribution, discussing the impact of sample size and parameter configurations on method selection.

Additional research by Alduais (2021) and Zheng et al. (2024) further emphasizes the need for robust estimation techniques. Although these studies highlight the importance of refining estimation methods, comprehensive comparisons for the Kumaraswamy Weibull regression model remain scarce, a gap this paper aims to fill. Accordingly, we systematically examine MLE, OLS, WLS, CVM, and AD methods, assessing their performance through information criteria (AIC, BIC, and HQIC).

Pascoa et al. (2013) compared maximum likelihood estimation and a Bayesian approach for regression analysis with censored data as a foundation for parameter estimation of Kumaraswamy-type distributions. These and other recent studies show that the research interest in optimizing estimation methods is still active. However, the comparison of comprehensive comparisons for the Kumaraswamy Weibull regression model is limited, and this is a research gap that this study aims to fill. Based on this foundation, we thoroughly investigate five estimation techniques for five estimation techniques: maximum likelihood estimation, ordinary least squares, weighted least squares, Cramér-von Mises, and Anderson-Darling. The effectiveness of these methods is tested using financial data analysis focusing on economic value-added modeling. The performance of these estimation methods is compared using AIC, BIC, and HQIC criteria.

The main contributions of this study are: (1) An empirical comparison of five distinct estimation methods for the Kumaraswamy Weibull regression model. (2) An application to economic value-added data, highlighting practical implications in financial analysis. (3) Evidence of the Kumaraswamy Weibull model's superiority over the standard Weibull model for modeling complex financial distributions. (4) Practical guidelines for selecting the most appropriate estimation method based on empirical criteria.

The remainder of this paper is organized as follows. Section 2 introduces the Kumaraswamy Weibull distribution along with its properties and mathematical formulation. Section 3 develops the regression model framework. Section 4 details the five estimation methods under consideration. Section 5 applies these methods to economic value-added data, including model validation and comparative analyses. Finally, Section 6 concludes the study and outlines future research directions.

2. The Kumaraswamy Weibull Distribution

2.1 Definition and Properties

The density function of the Kumaraswamy Weibull distribution (Cordeiro et al., 2010) is given by:

$$f(x) = \frac{ab\lambda}{\beta^{\lambda}} \; x^{\lambda-1} \, e^{-(\frac{x}{\beta})^{\lambda}} \left[1 - e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right]^{a-1} \left\{1 - \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right)^{a}\right\}^{b-1} \tag{1}$$

Where $x \geq 0$ and shape parameters satisfy the conditions $a > 0$ and b > 0.

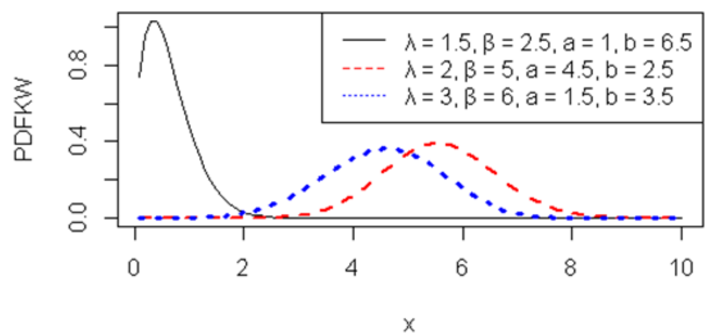

Figure 1. The probability density function for Kumaraswamy Weibull distribution with different parameter values

The density function demonstrates the distribution's flexibility in modeling various data patterns. For parameter values λ =1.5, β = 2.5, a =1, and b = 6.5, the distribution exhibits a pronounced peak near zero followed by a rapid decay. When parameters are set to λ =2, β = 5, a = 4.5, and b = 2.5, the distribution takes on a more symmetric bell-shaped form. The distribution becomes skewed with parameters λ = 3, β = 6, a = 1.5, and b = 3.5, illustrating how different parameter combinations can capture various shapes in the underlying data. As demonstrated in Figure 1, these diverse forms make the Kumaraswamy Weibull distribution particularly valuable for modeling financial data, which often exhibits complex patterns that simpler distributions cannot adequately capture.

2.2 Cumulative Distribution and Survival Functions

The cumulative distribution function is expressed as:

$$\mathrm{F(x)} = 1 - \left\{1 - \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right)^{a}\right\}^{\mathrm{b}} \tag{2}$$

for x≥0. This function describes the probability that the random variable takes a value less than or equal to x.

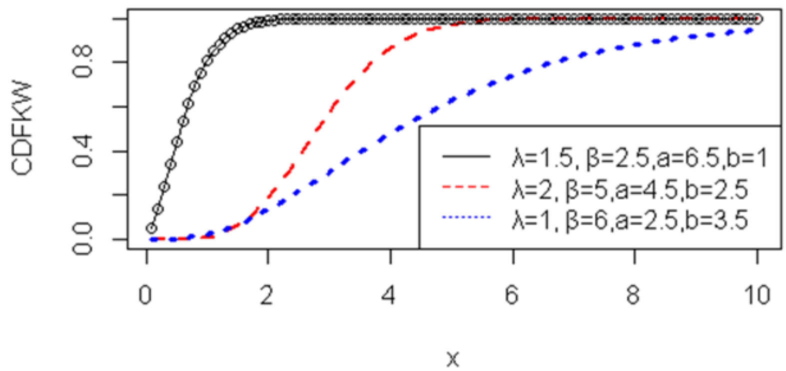

Figure 2. Cumulative Distribution Function of Kumaraswamy Weibull Distribution with Different Parameter Values

Figure 2 illustrates how parameter values affect the rate at which the CDF approaches 1. The configuration λ = 1.5, β = 2.5, a = 6.5, and b = 1 shows rapid convergence to 1, reaching near-complete probability mass by x = 2. The configurations λ = 2, β = 5, a = 4.5, b = 2.5 and λ = 1, β = 6, a = 2.5, b = 3.5 show more gradual increases, with the latter maintaining substantial probability mass beyond x =10, demonstrating the distribution's ability to model various tail behaviors. This flexibility in the cumulative distribution function further emphasizes the model's capacity to represent different risk patterns in financial data. The survival function, which represents the probability of an event occurring after time x, is:

$$S(x) = \left\{1 - \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\lambda}}\right)^{a}\right\}^{b} \qquad (3)$$

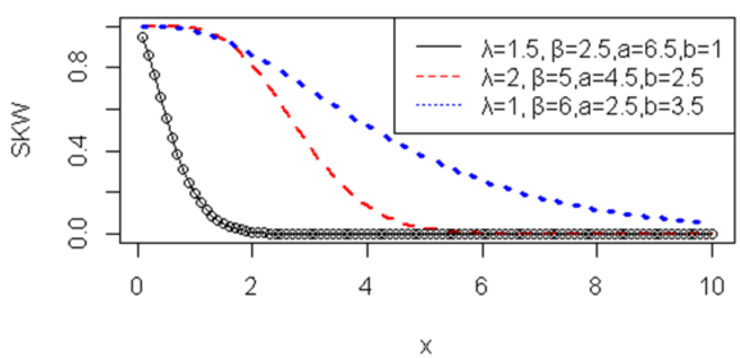

Figure 3. Survival Function of Kumaraswamy Weibull Distribution for Different Parameter Values

As shown in Figure 3, the survival function exhibits distinct patterns under different parameter configurations. With λ= 1.5, β = 2.5, a = 6.5, b = 1, survival probability drops rapidly, approaching zero by x = 2. The configuration λ = 2, β = 5, a = 4.5, and b = 2.5 shows a moderate decline, with approximately 50% survival at x = 4. The most gradual decline occurs with λ = 1, β = 6, a = 2.5, b=3.5, maintaining substantial survival probability beyond x = 8. These varying survival patterns highlight the distribution's versatility in modeling different durations until an event occurs, which is particularly useful for financial analyses involving time-to-event data.

2.3 Hazard Function and Moments

The hazard function, in terms of the probability density function f(x), is derived as:

$$h(x) = \frac{a\mathrm{b}\lambda\, x^{\lambda-1}\left[1-e^{-(\frac{x}{\beta})^{\lambda}}\right]^{a-1}}{\beta^{\lambda}\, e^{-(\frac{x}{\beta})^{\lambda}}\left\{1-\left(1-e^{-(\frac{x}{\beta})^{\lambda}}\right)^{a}\right\}} \tag{4}$$

The nth moment about zero for the Kumaraswamy Weibull distribution is given by:

$$\mu_n' = \frac{\psi\beta^{n+\lambda}}{\lambda(i+r+1)^{\frac{\lambda+n}{\lambda}}}\,\Gamma\left(\frac{n}{\lambda}+1\right) \tag{5}$$

Where: $\psi = \frac{W_i \mathrm{q}_{k,r}\, a\mathrm{b}\lambda}{\beta^{\lambda}}$, $\mathrm{q}_{k,r=} \sum_{k=0}^{\infty}(-1)^k\binom{b-1}{k}\sum_{r=0}^{\infty}(-1)^r\binom{ak}{r}$, $W_i = \sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}$

The mean (first moment) is thus:

$$\mu_1' = \frac{\psi\beta^{1+\lambda}}{\lambda(i+r+1)^{\frac{\lambda+1}{\lambda}}}\,\Gamma\left(\frac{1}{\lambda}+1\right) \tag{6}$$

and the variance can be computed using:

$$\sigma^2 = \mu_2 = \frac{\psi\beta^{2+\lambda}}{\lambda(i+r+1)^{\frac{\lambda+2}{\lambda}}}\,\Gamma\left(\frac{2}{\lambda}+1\right) - \left(\frac{\psi\beta^{1+\lambda}}{\lambda(i+r+1)^{\frac{1}{\lambda}+1}}\,\Gamma\left(\frac{1}{\lambda}+1\right)\right)^2 \tag{7}$$

where: $\frac{\lambda\beta^{2+\lambda}}{\psi(i+r+1)^{\frac{\lambda+2}{\lambda}}}\,\Gamma\left(\frac{2}{\lambda}+1\right) > \left(\frac{\beta^{1+\lambda}}{(i+r+1)^{\frac{1}{\lambda}+1}}\,\Gamma\left(\frac{1}{\lambda}+1\right)\right)^2$

These functions provide a comprehensive framework for modeling various data patterns, including those with increasing, decreasing, and non-monotonic hazard rates.

3. Kumaraswamy Weibull Regression Model

Let X be a random variable following the Kumaraswamy Weibull density function given in Section 2.1. If we define y $= log(x), \mu = log(\beta), and\ \lambda = \frac{1}{\sigma}$, the density function of y can be expressed as:

$$f(y) = \frac{ab}{\sigma}.\, e^{(\frac{y-\mu}{\sigma})}\, e^{-e^{(\frac{y-\mu}{\sigma})}}\left[1-e^{-e^{(\frac{y-\mu}{\sigma})}}\right]^{a-1}\left\{1-\left(1-e^{-e^{(\frac{y-\mu}{\sigma})}}\right)^{a}\right\}^{b-1} \tag{8}$$

For y ∈ ℝ. This represents the Log Kumaraswamy Weibull distribution, denoted as Y ~ LKW (a,b,σ,μ) , where μ ∈ ℝ is the location parameter, σ > 0 is the scale parameter that determines the distribution's spread, and a, b > 0 are shape parameters that control the distribution's form. The cumulative distribution function and survival function are:

$$F(y) = 1 - \left\{1 - \left(1 - e^{-e^{(\frac{y-\mu}{\sigma})}}\right)^{a}\right\}^{b} \quad (9)$$

$$S(y) = \left\{1 - \left(1 - e^{-e^{(\frac{y-\mu}{\sigma})}}\right)^{a}\right\}^{b} \quad (10)$$

For the regression model, we consider the following framework:

$$Y_i = X_i^T \beta + \sigma_i Z_i \quad , i = 1,2,3, \dots, n \quad (11)$$

where:

- Y_i is the response variable following the Log Kumaraswamy Weibull distribution
- ${X_i}^T = (1, x_{i1}, x_{i2}, \dots, x_{ip})$ is a vector of p+1 covariates
- $\beta = (\beta_0, \beta_1, \dots \dots \dots . \beta_p)^T$ is a vector of p+1 unknown regression coefficients
- $\sigma_i Z_i$ is the error term

Using the link function:

$$\mu = X^T \beta = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p \quad (12)$$

We can express the regression model of log Kumaraswamy Weibull distribution as:
$\log \mu(y_i) = X_i^T \beta; i = 1,2,3, \dots, n$
Therefore, the log Kumaraswamy Weibull distribution regression model can be written as a linear log location-scale regression model.

4. Parameter Estimation Methods

This section presents five distinct estimation methods for the Kumaraswamy Weibull regression model parameters, each approaching the estimation problem from a different theoretical perspective.
The comparative evaluation of these methods constitutes a central contribution of this research.
The five estimation methods examined are:

1. **Maximum likelihood estimation (MLE)**: A parametric approach that maximizes the probability of observing the specific data under the assumed distribution. MLE has theoretical advantages, including asymptotic efficiency and consistency. The log-likelihood function is given by: $ln\, L(\theta) = \sum[\delta_i \, ln\, f(y_i)] + \sum[(1-\delta_i) ln\, S(y_i)]$ where δ_i indicates censoring. Parameter estimates are obtained via numerical solutions of the resulting system of equations.
2. **Ordinary least squares (OLS)**: A distance minimization approach that estimates parameters by fitting a line through the points such that the vertical distance, or residual, between the observed data and the fitted line, is minimized, also known as the sum of squared errors. The least squares method has a long history in statistical modeling (Plackett 1972, Stigler 1981).
3. **Weighted least squares (WLS)**: An extension of OLS that uses weights to the observations based on the variance of each observation such that it may be more efficient when the data has heteroscedasticity. This approach is based on the work of Aitken (1936) for the linear combination of observations.
4. **Cramér-von Mises (CVM)**: A minimum distance method that uses integrated squared difference as a measure of distance to calculate the difference between the empirical and theoretical distribution functions, and it does so uniformly across the distribution.

The theoretical results for minimum distance methods in distribution fitting were proved by Wolfowitz (1957).

5. **Anderson-Darling (AD)**: A weighted variant of CVM that puts more weight on the observations in the distribution's tails, which can be helpful for data with necessary tail behavior.

The comparative performance of these estimation methods will be assessed using three well-established information criteria:

- **Akaike Information Criterion (AIC)**: AIC $= 2k - 2ln(L)$, where k is the number of parameters, and L is the maximized value of the likelihood function. AIC is a tradeoff between the fit of the model and complexity.
- **Bayesian Information Criterion (BIC)**: BIC $= k \times ln(n) - 2ln(L)$, where n is the sample size. BIC is more conservative than AIC and penalizes complex models more severely, especially when working with large sample sizes.
- **Hannan-Quinn Information Criterion (HQIC)**: HQIC $= 2k \times ln(ln(n)) - 2ln(L)$. HQIC presents a penalty for complexity that lies between AIC and BIC.

Models with smaller AIC, BIC, or HQIC values are preferred because they provide reasonably good explanatory power without unnecessary complexity.

4.1 Maximum Likelihood Estimation (MLE)

The log-likelihood function for the Kumaraswamy Weibull regression model is:

$$lnL(\theta) = \sum_{i=1}^{n} \delta_i Ln(f(y_i)) + \sum_{i=1}^{n} (1-\delta_i) ln(S(y_i)) \tag{13}$$

Where θ represents the parameter vector $(\beta_0, \beta_1, \ldots\ldots, \beta_p, \sigma, a, b)$ and δ_i indicates whether the ith observation is censored (0) or uncensored (1).
By substituting the density and survival functions, we obtain:

$$lnL(\theta) = n\delta_i Ln\left(\frac{ab}{\sigma}\right) + \sum_{i=1}^{n} \delta_i \left[\left(\frac{y_i - X_i^T\beta}{\sigma}\right) - e^{\left(\frac{y_i - X_i^T\beta}{\sigma}\right)}\right]$$

$$+(a-1)\sum_{i=1}^{n} \delta_i Ln\left[1 - e^{-e^{\left(\frac{y_i - X_i^T\beta}{\sigma}\right)}}\right] + \sum_{i=1}^{n} (b-\delta_i) ln\left[1 - \left(1 - e^{-e^{\left(\frac{y_i - X_i^T\beta}{\sigma}\right)}}\right)^a\right] \tag{14}$$

The parameter estimates are obtained by setting the partial derivatives of the log-likelihood function concerning each parameter equal to zero and solving the resulting system of equations using numerical methods.

4.2 Ordinary Least Squares (OLS)

The OLS estimators for the Kumaraswamy Weibull distribution are obtained by minimizing:

$$S = \sum_{i=1}^{n} \left[F(y_{(i)}) - \frac{i}{n+1}\right]^2 \tag{15}$$

For the Kumaraswamy Weibull distribution with multiple covariates, this becomes:

$$S = \sum_{i=1}^{n}\left[1 - \left\{1 - \left(1 - e^{-e^{(\frac{y_i - X_i^T\beta}{\sigma})}}\right)^{a}\right\}^{b} - \frac{i}{n+1}\right]^2 \quad (16)$$

Parameter estimates are obtained by setting the partial derivatives for each parameter equal to zero and solving the resulting system of equations. Least-squares approaches have been successfully applied to estimate parameters in various distribution systems (Swain et al., 1988).

4.3 Weighted Least Squares (WLS)

The WLS approach addresses heteroscedasticity by assigning weights to observations. The estimators are obtained by minimizing the following:

$$W = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n-i+1)i}(F(y_{(i)}) - \frac{i}{n+1})^2 \quad (17)$$

For the Kumaraswamy Weibull distribution with multiple covariates:

$$W = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n-i+1)i}(1 - \left\{1 - \left(1 - e^{-e^{(\frac{y_i - X_i^T\beta}{\sigma})}}\right)^{a}\right\}^{b} - \frac{i}{n+1})^2 \quad (18)$$

4.4 Cramér-von Mises (CVM)

The CVM estimators minimize:

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n}\left(1 - \left\{1 - \left(1 - e^{-e^{(\frac{y_i - X_i^T\beta}{\sigma})}}\right)^{a}\right\}^{b} - \frac{2i-1}{2n}\right)^2 \quad (19)$$

4.5 Anderson-Darling (AD)

The AD estimators are obtained by minimizing:

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i - 1)\log\left\{\left(1 - \left\{1 - \left(1 - e^{-e^{\left(\frac{y_i - X_i^T\beta}{\sigma}\right)}}\right)^{a}\right\}^{b}\right) \times \left(\left\{1 - \left(1 - e^{-e^{(\frac{y_{(n-i+1)} - X_{(n-i+1)}^T\times\beta}{\sigma})}}\right)^{a}\right\}^{b}\right)\right\} \quad (20)$$

5. Application to Economic Value-Added Data

5.1 Data description

The dataset contains economic value-added measurements from five companies across 24 quarterly periods (2016-2021). The data structure includes:

Dependent variable:

- Economic value added - This variable represents the economic profit of the firms

Independent variables:

- Average collection period - This variable represents the average time taken for a company to collect payments from its customers, measured in days
- Firm size - This variable represents the size of the firm, measured by the natural logarithm of total assets
- Leverage - This variable represents the firm's level of debt about its assets, calculated as the ratio of total debt to total assets.

Based on financial theory, we anticipate that Economic Value Added will be positively associated with Firm Size due to economies of scale and negatively associated with Average Collection Period and Leverage due to their effects on capital efficiency and financial risk, respectively.

5.2 Model Validation

5.2.1 Goodness-of-Fit Test of Real Data

To assess the appropriate distribution for modeling, we compared the goodness-of-fit statistics between Kumaraswamy Weibull and standard Weibull distributions.

Table 1. Goodness-of-Fit Test Results

The goodness of fit statistics	Kumaraswamy Weibull	Weibull	p-value (KW)	p-value (Weibull)
Kolmogorov- Smirnov	0.12356	0.20952	0.196	0.185
Cramer- von statistic	0.34865	1.45120	-	-
Anderson- darling	2.13240	8.42470	-	-

Table 1 comprehensively compares goodness-of-fit statistics between the Kumaraswamy Weibull and standard Weibull distributions. The Kumaraswamy Weibull distribution consistently outperforms the Weibull distribution across all three tests. Lower values indicate a better fit to the real data, suggesting that the Kumaraswamy Weibull distribution is more suitable for modeling the economic value-added data. The p-value for the Kumaraswamy Weibull model (0.196) exceeds the conventional significance level of 0.05, indicating that we cannot reject the null hypothesis that the data follows this distribution. Similarly, the p-value for the Weibull model (0.185) is also above the significance threshold, though its Kolmogorov- Smirnov statistic (0.20952) is substantially higher than that of the Kumaraswamy Weibull model (0.12356). This statistical evidence, along with the significantly lower Cramer-von Mises and Anderson-Darling statistics, supports our choice of the Kumaraswamy Weibull distribution as the appropriate model for this financial dataset.

5.2.2 Histogram Fitting

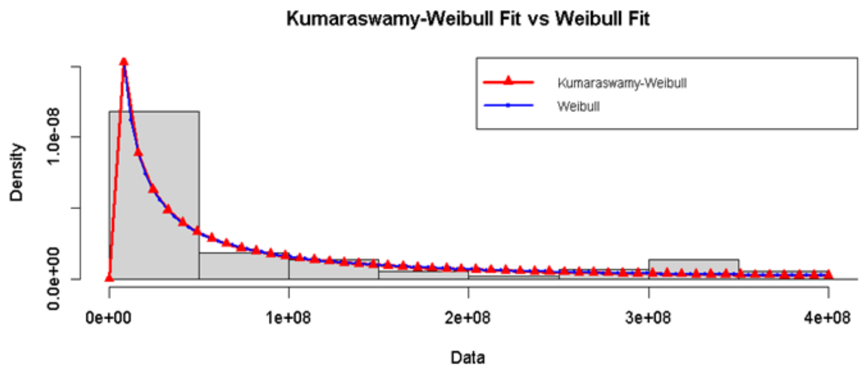

Figure 4. Fitting Data with Kumaraswamy Weibull and Weibull Distributions

Figure 4 illustrates the empirical distribution of the data alongside the fitted Kumaraswamy Weibull and standard Weibull probability density functions. Both distributions capture the data distribution's general shape, exhibiting positive skewness with a concentration of values near zero and a long right tail. The Kumaraswamy Weibull distribution provides a slightly better fit, particularly at the peak density and tail region. The superior fit of the Kumaraswamy Weibull distribution is especially apparent in the right tail, where the standard Weibull distribution underestimates the probability density. While both models adequately represent the data distribution, the Kumaraswamy Weibull distribution's additional parameters allow for more precise modeling of the data's characteristics, making it particularly valuable for financial data that often exhibit complex distribution patterns.

5.2.3 Analysis of Variance

Table 2. Kumaraswamy Weibull ANOVA Results

Source	DF	SS	MS	F	p-value
Model	3	9.6423109	3.21410365	1044.374	9.67e-84
Residual	116	0.3569946	0.00307754		
Total	119	9.9993056			

Table 3. Weibull ANOVA Results

Source	DF	SS	MS	F	p-value
Model	3	8.450924	2.81697452	211.039	8.24e-47
Residual	116	1.548382	0.01334812		
Total	119	9.999306			

The ANOVA results (Tables 2 and 3) reveal that both models are statistically significant. However, the Kumaraswamy Weibull model demonstrates a superior fit with a substantially higher F-statistic (1044.374 vs. 211.039) and lower p-value (9.67e-84 vs. 8.24e-47). The dramatically higher F-value for the Kumaraswamy Weibull model indicates that its explanatory power is approximately five times greater than the standard Weibull model. Additionally, the Kumaraswamy Weibull model explains more variance, as evidenced by its more significant model sum of squares (9.6423109 vs. 8.450924) and smaller residual sum of squares (0.3569946 vs. 1.548382). This means the Kumaraswamy Weibull model leaves less unexplained variation, further supporting its selection as the more appropriate model for this economic value-added data.

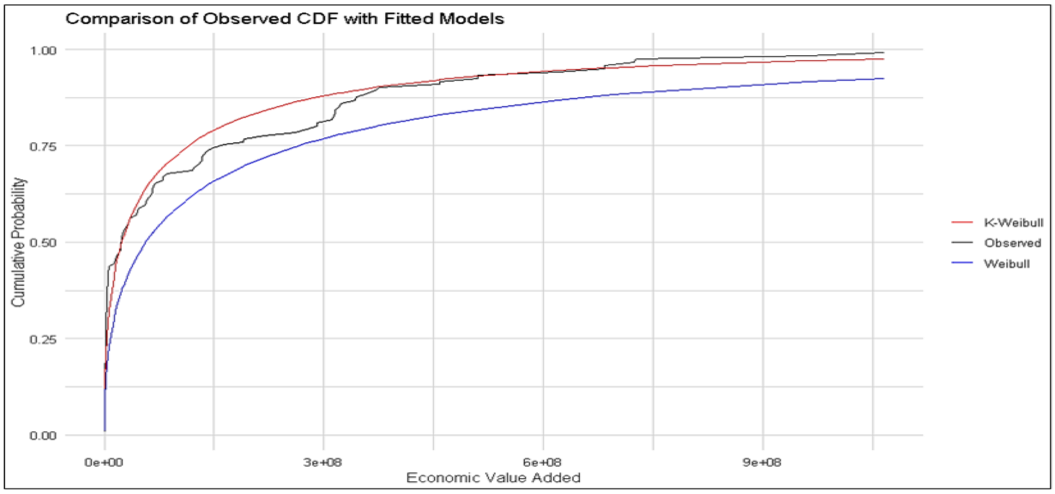

Figure 5. Comparison of CDF with Fitted Models

Figure 5 demonstrates the superior fit of the Kumaraswamy Weibull model to the observed cumulative distribution function compared to the standard Weibull model. While both models perform adequately in the lower range of values, the Kumaraswamy Weibull model maintains accuracy throughout the entire range, particularly in the middle and upper regions where the standard Weibull model significantly underestimates cumulative probability. This visual evidence further supports the numerical findings from the goodness-of-fit tests and ANOVA analyses, confirming the Kumaraswamy Weibull distribution's superior ability to capture complex patterns in the economic value-added data.

5.3 Comparison of estimation methods

Five estimation methods based on information criteria were compared to determine which approach provides the best parameter estimates for the Kumaraswamy Weibull regression model.

Table 4. Parameter Estimates from Different Estimation Methods

	MLE	OLS	WLSE	CVM	AD
$\hat{a}$	0.3709	0.6108	0.7113	0.5621	0.2893
$\hat{b}$	783.4944	2.7097	1.9335	3.5682	1.8390
$\widehat{\beta_0}$	4.926352	18.203455	18.709562	18.051014	15.799424
$\widehat{\beta_1}$	-0.006491	0.060773	0.004167	0.046024	0.004388
$\widehat{\beta_2}$	0.822918	-0.244148	-0.010471	-0.173840	-0.053979
$\widehat{\beta_3}$	-1.4511	-0.2672	-0.3083	-0.2442	1.8153

Table 4 displays the parameter estimates obtained using the five different estimation methods. Notable differences can be observed across methods, particularly for the shape parameter $\hat{b}$, where MLE produces a substantially higher estimate (783.4944) than other methods (ranging from 1.8390 to 3.5682). Similarly, the regression coefficients show considerable variation, with MLE suggesting a positive relationship between firm size ($\widehat{\beta_2}$ = 0.822918) and economic value added, while other methods indicate a negative relationship. These disparities highlight the method selection's crucial role in parameter estimation and subsequent model interpretation.

Table 5. Model Selection Criteria for Different Estimation Methods

Criterion	MLE	OLS	WLSE	CVM	AD
AIC	4295.875	4536.706	4565.007	4530.103	4511.800
BIC	4332.112	4572.943	4601.244	4566.340	4548.037
HQIC	4310.591	4551.422	4579.723	4544.819	4526.516

While the log-likelihood function is presented in its general form that can accommodate censored data, no censored observations are contained in the economic value-added dataset. This means the indicator variable δ_i equals 1 for all observations in the analysis. The model selection criteria values for each estimation method are provided in Table 5. The lowest AIC (4295.875), BIC (4332.112), and HQIC (4310.591) values among all five methods are consistently yielded by maximum likelihood estimation, indicating superior performance in terms of balancing model fit and complexity. Second place is ranked by Anderson-Darling estimation, while the highest values across all three criteria are consistently shown by weighted least squares, suggesting it may be less suitable for this dataset. Strong evidence for selecting MLE as the optimal estimation approach for the Kumaraswamy Weibull regression model when applied to economic value-added data is provided by the substantial difference in information criteria values between MLE and other methods (differences of over 200 points in some cases).

Based on this analysis, Maximum Likelihood Estimation emerges as the superior method for estimating Kumaraswamy Weibull regression parameters for the Economic Value-Added data. Given its optimal performance across all information criteria, MLE will be used for subsequent model comparisons.

5.4 Comparison between Kumaraswamy Weibull and Weibull Regression Models

Using the MLE method, the performance of Kumaraswamy Weibull regression was compared against standard Weibull regression.

Table 6. Model Comparison Results

Regression model	$\widehat{\beta_0}$	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{\beta_3}$	HQIC	BIC	AIC	R^2
Kumaraswamy Weibull	4.926	-0.006	0.823	-1.451	4310.5	4332.1	4295	0.96
Weibull	0.115	-0.004	0.995	-0.154	4572.2	4580.5	4566	0.84

Based on the regression results, the Kumaraswamy Weibull regression model can be expressed as:

$log\hat{\mu}_i$(Economic Value Added) = 4.926 - 0.006 (Average Collection Period) + 0.823(Firm Size) -1.451 (Leverage)

This equation demonstrates that Economic Value Added is negatively associated with Average Collection Period and Leverage while positively associated with Firm Size. The coefficients indicate that a one-unit increase in firm size is associated with a substantial increase in added economic value, while increased leverage significantly reduces it. These findings align with financial theory: larger firms often benefit from economies of scale, enhancing their ability to generate value beyond capital costs; higher leverage increases financial risk and interest expenses, reducing economic profit; and more extended collection periods tie up working capital, diminishing a firm's ability to generate additional value. These relationships provide actionable insights for financial managers seeking to improve their firms' economic value creation.

5.5 Limitations and Future Research Directions

As there are some limitations in this study, they should be acknowledged. First, our analysis was performed on data from a single sector, which may lead to a lack of generalization of the results. Future work should subsume this comparative analysis to other industry sectors, including manufacturing, healthcare, or technology, to check whether the methodological findings of this study are supported in other settings.

Moreover, examining how sample size influences the comparative effectiveness of these estimation procedures would be helpful for researchers and practitioners working with various-sized datasets. Other studies could also examine the behavior of these methods with respect to different parameters and data properties, e.g., the presence of heavy tails or outliers in the data.

Another interesting direction for future work would be to contrast the Kumaraswamy Weibull distribution with other flexible distributions for financial data. The Kumaraswamy family has been generalized to other base distributions, such as the log-logistic distribution (De Santana et al., 2012), log exponential power distribution (Korkmaz et al., 2021), and the extended Pareto distribution (Alshanbari et al., 2021). These comparisons would also help expand the knowledge of which distributional forms are suitable for which financial metrics and contexts, thus supporting the importance of this class of models.

6. Conclusion

This research makes a significant methodological contribution by providing the first comprehensive, empirical comparison of five distinct parameter estimation methods for the Kumaraswamy Weibull regression model. A rigorous analysis using economic value-added data establishes Maximum Likelihood Estimation as the superior approach, consistently outperforming Ordinary Least Squares, Weighted Least Squares, Cramér-Von Mises, and Anderson-Darling methods across all information criteria (AIC, BIC, and HQIC). This finding aligns with theoretical expectations regarding MLE's asymptotic efficiency while providing empirical validation in the specific context of economic value-added modeling.

Furthermore, the comparative analysis between the Kumaraswamy Weibull and standard Weibull regression models demonstrates the substantial advantages of the more flexible four-parameter distribution. The Kumaraswamy Weibull model achieved a significantly better fit ($R^2 = 0.96\ vs.\ 0.84$), lower information criteria values, and superior performance in goodness-of-fit tests. These empirical results validate the theoretical advantages of the Kumaraswamy Weibull distribution's ability to model complex data patterns, particularly for financial data characterized by asymmetry and heavy tails.

From a practical perspective, the findings provide financial analysts with methodological guidance and actionable insights. The regression results reveal that economic value added is positively associated with firm size ($\beta_2 = 0.823$) and negatively associated with both leverage ($\beta_3 = -1.451$) and collection periods ($\beta_1 = -0.006$). These statistically significant relationships align with theoretical expectations from financial economics and provide quantitative evidence for strategies to enhance economic value creation. The parameter estimates' magnitudes offer precise guidance on the relative importance of these factors, with leverage having the most substantial impact on economic value added.

While the in-depth analysis focused on economic value-added data from a specific sector, allowing for detailed parameter estimation and model comparison, future research should extend these methods to other financial contexts and industry sectors. The strong theoretical foundation of the Kumaraswamy Weibull distribution, combined with the empirical validation of MLE as the optimal estimation approach, provides a robust framework for such extensions.

This study bridges an essential gap between statistical theory and applied financial modeling by demonstrating how advanced distributional forms and estimation techniques can significantly improve model fit and predictive accuracy in economic value-added analysis. The findings contribute to both the statistical literature on parameter estimation and the financial literature on economic value modeling, providing a methodologically sound approach for researchers and practitioners working with complex financial data.

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