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Bayesian Inference on Residual Tsallis Entropy of the Inverse Weibull Model: A Progressive Type I Censoring Approach

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Abstract

Lately, there has been a great deal of interest in the process of measuring uncertainty about probability distributions. Shannon (1948) introduced the Shannon entropy measure to the field of information theory. Based on Ebrahimi (1996), the residual version of entropy functions is introduced. Throughout this work, the estimation of residual Tsallis (RT) entropy for the inverse Weibull (IW) distribution under the progressive Censoring Type-I approach is discussed. Non-Bayesian and Bayesian inference methods are used to estimate the RT entropy of the IW distribution. The RT Bayesian estimators are evaluated under non-informative and informative priors based on linear exponential loss functions and squared error. A Monto Carlo simulation study and an illustration using actual data sets were conducted to assess the estimators' performance. From simulated outcomes, RT maximum likelihood and Bayesian estimators under progressive censoring Type-I perform well as the sample size grows. Additionally, Bayesian estimators of RT entropy measure under the LINEX-II loss function are superior to other competing estimators.

1. Introduction

Studying uncertainty measures, known as entropy, is of great importance in probability distributions in recent years. Entropy quantifies the average of information included in a random variable. Reduced information in the sample is indicated by a higher entropy value. The measurement of entropy is a significant concept in numerous fields, including computer science, statistics, information technology, economic, physical, chemical and biological phenomena.

Tsallis (1988) defined the concept of entropy as a measure of uncertainty in a random observation. It has been widely employed as a basis for generalizing the standard statistical mechanics and used in quantum fields such as communication protocol systems and correlations (see Renner et al. (2005), Lévay et al. (2005)). For a continuous random variable with a cumulative distribution function (CDF), namely, K(x), and a probability density function (PDF), say, k(x), the Tsallis (T) entropy of order α is provided via

$$T(\alpha) = \frac{1}{\alpha-1}\left[1-\int_0^\infty k_x^{\alpha}(x)\,dx\right] \qquad ,\alpha>0,\alpha\neq 1 \qquad (1)$$

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T entropy of order α is a generalization of type α of the Shannon entropy, first introduced by Shannon (1948). While Shannon entropy may be negative for some distributions, T entropy can always be made non-negative by choosing an appropriate value of α.

The estimation of entropy measures for various statistical distributions has been the subject of discussion among numerous investigators (see Abo-Eleneen (2011), Al-Babtain et al. (2021)), and Cohen (1963) recently examined in reliability and survival research. In some cases, the amount of time spent studying has been regarded as a key variable of interest in numerous fields; thus, the information measures are functions of time. Based on this idea, Ebrahimi (1996) introduced the residual entropy function, which measures the uncertainty about the remaining lifetime of a system if it is working at time t, called the residual Shannon entropy. A residual version of Renyi's entropy is due to Ibrahimi and Sankaran (2005). For T be a non-negative random variable representing the lifetime of a component with $K(t) = P(X \le t)$, Nanda and Paul (2006) defined a residual version of Tsallis entropy (RT) entropy as:

$$RT(\alpha,t) = (\alpha-1)^{-1}\left[1-\left(\bar{K}_X(t)\right)^{-1}\int_t^{\infty} k_X^{\alpha}(x)\,dx\right] \quad , \quad \text{where } \alpha>0, \alpha\neq 1 \qquad (2)$$

where $\bar{K}(.) = 1-K(.)$ is the survival function. For more detailed studies on residual versions of entropy see Song (2001), Asadi et al. (2005, 2006), Baratpour et al. (2008), Zarezadeh and Asadi (2010), Li and Zhang (2011), Sunoj and Sankaran (2012), and Fashandi and Ahmadi (2012).

The failure times of censored data are usually not observed for all units. This data contributes valuable information and should not be omitted from the analysis. There are various censoring schemes, such as single, double, random, hybrid, and progressive censoring.

The conventional Type-I and Type- II censoring schemes do not have the flexibility of the allowing removal of units at points other than the terminal point of the experiment. Because of this lack of flexibility, intermediate removal may be desirable for some tests. These reasons lead reliability practitioners and theoreticians directly to the area of progressive censoring schemes.

The idea behind progressive censoring samples arises when at various stages of an experiment; some items are removed from the life test throughout the test, and some, though not all, of the surviving specimens remaining are eliminated from further observations. Sample specimens remaining after each stage of censoring are continued under observation until failure or until a subsequent stage of censoring (Balakrishnan and Cramer (2010), for more details; (see Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014)).

The items removed from the experiment at

- A prefixed time of censoring is referred to as progressive censoring Type-I.
- The failure time of the items is referred to as progressive censoring Type-II.

Progressive censoring Type-I (PCT_I) occurs when n units are put on life test at time zero and R_j items are removed from the survivor items at the predetermined time of censoring T_j , $j=1, 2,...,m$ where m is the number of stages on the test, $T_j > T_{j-1}$ and $r = n - \sum_{j=1}^{m} R_j$ .The experiment is finished when the time T_m elapses and the remaining R_m surviving units are all removed, so $R_m = n - \left[\sum_{j=1}^{m-1} R_j + r \right]$, as a schematic illustration is depicted in Figure 1. Cohen (1963) introduced the likelihood function in this case as follows:

$$L = C \prod_{i=1}^{r} k(x_{(i)}) \prod_{j=1}^{m} \left[1 - K(T_j) \right]^{R_j}, \tag{3}$$

where C is a constant which independent of the parameters, $x_{(i)}$ represents the lifetime of the i^{th} order statistic and $k(.)$, $K(.)$ denote PDF and CDF of the underlying distribution respectively. For $R_1 = R_2 = ... = R_{m-1} = 0$, then $\mathrm{R}_m = n - r$, $\mathrm{PCT_I}$ reduces to a single censoring Type-I scheme (see Balakrishnan and Aggarwala (2000)).

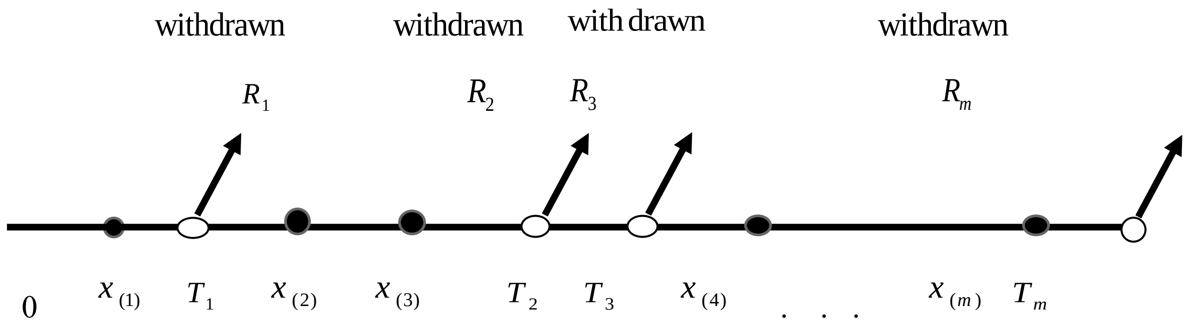

Figure 1. Progressive Type-I Censoring Scheme

Many researchers have investigated progressive censoring schemes; for instance, Musleh and Helu (2014) defined parameter estimation of IW distribution under progressively Type-II censoring utilizing both classical and Bayesian approaches. Cho et al. (2015) provided the entropy Bayesian estimators of Weibull distribution under a generalized progressive hybrid censoring scheme. Lee (2017) considered entropy estimation for the IW model using maximum likelihood (ML) and Bayesian estimation in the case of the generalized progressive hybrid censoring (GPHC) scheme. For the exponential distribution, Almohaimeed (2017) provided an exact expression for entropy information included in progressively hybrid censored data. Vishwakarma et al. (2018) addressed the IW parameter estimation procedure under progressive Type-II censored samples when removals follow the Beta-binomial probability law. Hassan and Zaky (2019) derived the Shannon entropy ML estimator of the IW distribution from multiple censored data. Algarni et al. (2021) estimated unknown parameters of the IW distribution employing progressive censored data Type-I. Ren and Hu (2023) obtained the statistical inferences for the IW distribution under a progressive Type-II censored sample.

A few texts have addressed the estimation of residual entropy in the case of $\mathrm{PCT_I}$. Considering this, our objective is to investigate both Bayesian and non-Bayesian estimation of RT entropy for the IW model in the case of $\mathrm{PCT_I}$ the scheme. The Bayesian estimators are evaluated under both non-informative and informative priors, utilizing linear exponential Loss functions and squared error.

This work is ordered as follows: In section 2, RT entropy of IW distribution. Section 3 provides a non-Bayesian estimation of the RT entropy of IW under $\mathrm{PCT_I}$ sample. Bayesian estimators of the RT entropy of IW utilizing different loss functions in the case $\mathrm{PCT_I}$ are presented in section 4. In Sections 5 and 6 respectively; applications to real data and Monto Carlo simulation studies are obtained. The article ends with some concluding remarks.

2. Residual Entropy of IW Model

The IW model is one of the most significant lifetime models. It can be used in a wide range of applications like reliability, life testing, engineering, and survival analysis. Moreover, the IW distribution has proven valuable in the reliability engineering discipline, effectively modeling various failure characteristics including infant mortality, useful life, and wear-out periods (see Khan et al. (2008) and Alkarni et al. (2020)). The CDF of the IW model takes the following form.

$$K(x)=e^{-\left(\frac{\omega}{x}\right)^{\upsilon}}, \qquad x>0,\omega,\upsilon>0 \tag{4}$$

where ω is a scale parameter and υ is a shape parameter. The RT entropy of the IW model can be derived by inserting (4) in (2) as follows:

$$RT(\alpha,t)=(\alpha-1)^{-1}\left[1-\left(1-e^{-\left(\frac{\omega}{t}\right)^{\upsilon}}\right)^{-1}\phi\gamma\left(\alpha+\frac{\alpha}{\upsilon}+\frac{1}{\upsilon}\ ,\alpha\ \omega^{\alpha}t^{-\upsilon}\right)\right] \tag{5}$$

where $\alpha>0,\alpha\neq1$, $\phi=\upsilon^{\alpha-1}\omega^{\alpha\omega}\left(\alpha\omega^{\upsilon}\right)^{-\alpha-\frac{\alpha}{\upsilon}-\frac{1}{\upsilon}}$, $\gamma(S\ ,x)=\int_0^x f^{S-1}e^{-f}\,df$ lower incomplete gamma.

Xu and Gui (2019) discussed entropy estimation for a two-parameter IW model under adaptive Type-II progressive hybrid censoring schemes. Elbiely (2019) introduced a new flexible extension of the IW model, and some important mathematical properties of the proposed model were derived along with a numerical analysis of the mean, variance, skewness, and kurtosis measures for the proposed model. Basheer (2019) provided an alpha power inverse Weibull distribution. Basheer et al. (2020) introduced Marshall-Olkin alpha power IW. Alkarni et al. (2020) proposed the new three-parameter Type-I half-logistic IW (TIHLIW) distribution, which generalizes the IW model. Jana and Bera (2020) established the existence and uniqueness of the Maximum likelihood (ML) estimators of the scale and shape parameters and derived Bayes estimators of the parameters under the entropy loss function. Sadiah and Abdallah (2023) estimated using PDFs such as the IW and Rayleigh, presenting an extensive mathematical approach for wind speed evaluation using the IW distribution.

3. Non-Bayesian Inference

Within this part, the non-Bayesian inference (NBI) known as the ML method is considered for the RT entropy of the IW distribution under $\mathrm{PCT_I}$ Sample.

Maximum Likelihood Estimation for IW:

The likelihood function under PCT_{I} sample of the IW distribution is evaluated by substituting equation (4) into (3):

$$L(\upsilon,\omega)\propto\left(\upsilon\,\omega^{\upsilon}\right)^{r}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} \tag{6}$$

Take the logarithm of $L(\upsilon,\omega)$ to obtain the log-likelihood function of (6) obtained by

$$LL(\upsilon,\omega)\propto r\ln\upsilon+r\,\upsilon\ln\omega-(\upsilon+1)\sum_{i=1}^{r}\ln x_i-\sum_{i=1}^{r}\left(\frac{\omega}{x_i}\right)^{\upsilon}+\sum_{j=1}^{m}R_j\ln\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right] \tag{7}$$

First partial derivatives of (7) with respect to unknown parameters υ and ω are computed, respectively as follows:

$$\frac{\partial LL(\upsilon,\omega)}{\partial\upsilon}=\frac{r}{\upsilon}-r\ln\,\omega-\sum_{i=1}^{n}\ln\,x_i-\sum_{i=1}^{n}\left(\frac{\omega}{x_i}\right)^{\upsilon}\ln\left(\frac{\omega}{x_i}\right)+\sum_{j=1}^{m}R_j\left(\frac{1}{1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}}\right)e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\left(\frac{\omega}{T_j}\right)^{\upsilon}\ln\left(\frac{\omega}{T_j}\right) \tag{8}$$

$$\frac{\partial LL(\upsilon,\omega)}{\partial\omega}=\frac{r\,\upsilon}{\omega}-\upsilon\sum_{i=1}^{n}\left(\frac{\omega}{x_i}\right)^{\upsilon-1}\left(\frac{1}{x_i}\right)+\upsilon\sum_{j=1}^{m}R_j\left(\frac{1}{1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}}\right)e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\left(\frac{\omega}{T_j}\right)^{\upsilon-1}\left(\frac{1}{T_j}\right) \tag{9}$$

Equating $\left.\frac{\partial\,LL(\upsilon,\omega)}{\partial\upsilon}\right|_{\upsilon=\ddot{\upsilon}}=0$ and $\left.\frac{\partial\,LL(\upsilon,\omega)}{\partial\omega}\right|_{\omega=\ddot{\omega}}=0$,gives

$$\frac{r}{\ddot{\upsilon}}-r\ \ln\,\ddot{\omega}-\sum_{i=1}^{n}\ln\,x_i+\sum_{i=1}^{n}\left(\frac{\ddot{\omega}}{x_i}\right)^{\ddot{\upsilon}}.\ln\left(\frac{\ddot{\omega}}{x_i}\right)+\sum_{j=1}^{m}R_j\left(\frac{1}{1-e^{-\left(\frac{\ddot{\omega}}{T_j}\right)^{\ddot{\upsilon}}}}\right)e^{-\left(\frac{\ddot{\omega}}{T_j}\right)^{\ddot{\upsilon}}}\left(\frac{\ddot{\omega}}{T_j}\right)^{\ddot{\upsilon}}\ln\left(\frac{\ddot{\omega}}{T_j}\right)=0,$$

$$\frac{r\,\ddot{\upsilon}}{\ddot{\omega}}-\ddot{\upsilon}\sum_{i=1}^{n}\left(\frac{\ddot{\omega}}{x_i}\right)^{\ddot{\upsilon}-1}\left(\frac{1}{x_i}\right)+\ddot{\upsilon}\sum_{i=1}^{m}R_j\left(\frac{1}{1-e^{-\left(\frac{\ddot{\omega}}{T_j}\right)^{\ddot{\upsilon}}}}\right)e^{-\left(\frac{\ddot{\omega}}{T_j}\right)^{\ddot{\upsilon}}}\left(\frac{\ddot{\omega}}{T_j}\right)^{1-\ddot{\upsilon}}\left(\frac{1}{T_j}\right)=0,$$

It is observed that a closed-form solution to the above equations does not exist. So, a nonlinear optimization program will be employed to obtain the ML estimators numerically. Moreover, the ML estimators of RT entropy measure in equation (5) using the ML estimated parameters $\ddot{\upsilon}$ and $\ddot{\omega}$ respectively, can be computed by utilizing the invariance property as follows:

$$\ddot{R}T(\alpha,t)=(\alpha-1)^{-1}\left[1-\left(1-e^{-(\frac{\ddot{\omega}}{t})^{\ddot{\upsilon}}}\right)^{-1}\phi\gamma(\alpha+\frac{\alpha}{\ddot{\upsilon}}+\frac{1}{\ddot{\upsilon}}\,,\,\alpha\,\ddot{\omega}^{\alpha}\,t^{-\ddot{\upsilon}})\right] \tag{10}$$

4. Bayesian Inference

In this part, we considered the Bayesian Inference (BI) of the RT entropy for the IW distribution in the case of $\mathrm{PCT_I}$ the sample. Due to the important role of prior distribution in the BI, two types of priors will be considered: non-informative prior (NIP) and informative prior (IP) under SE and LINEX loss functions will be covered in this part. The Markov Chain Monte Carlo (MCMC) approach will be used to generate posterior samples by the Metropolis-Hastings (MH) algorithm to find the desired RT entropy under BI.

4.1 Residual entropy of $\mathrm{PCT_I}$ Estimators in the case of NIP

For NIP, BI estimators of RT entropy for the IW distribution are obtained in this subsection based on $\mathrm{PCT_I}$ the case of symmetric and asymmetric loss functions.

Assuming that the priors for parameters υ and ω, namely by, $\pi_1(\upsilon)$ and $\pi_2(\omega)$, respectively, have uniform distributions. The joint prior distribution for parameters υ and ω in the case of NIP is given as follows:

$$\pi_{1,2}(\upsilon,\omega/\underline{x}) \propto \frac{1}{\upsilon\,\omega} \tag{11}$$

Based on the likelihood function (6) and the joint prior distribution of parameters (11), the joint posterior density of unknown parameters, denoted by $\pi*_{1,2}(\upsilon,\omega/\underline{x})$ data can be written as follows:

$$\pi_{1,2}^{*}(\upsilon,\omega/\underline{x}) = \frac{L(\upsilon,\omega/x_1,x_2,...,x_n)\ \pi_{1,2}(\upsilon,\omega/\underline{x})}{\int_0^\infty\int_0^\infty L(\upsilon,\omega/x_1,x_2,...,x_n)\ \pi_{1,2}(\upsilon,\omega/\underline{x})\,d\upsilon\,d\omega}$$

$$\pi_{1,2}^{*}(\upsilon,\omega/\underline{x}) = \frac{\upsilon^{r-1}\,\omega^{\upsilon r-1}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\,e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j}}{\int_0^\infty\int_0^\infty \upsilon^{r-1}\,\omega^{\upsilon r-1}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\,e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\,d\omega}$$

then

$$\pi_{1,2}^{*}(\upsilon,\omega/\underline{x}) = c^{-1}.\,\upsilon^{r-1}\,\omega^{\upsilon r-1}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\,e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j}$$

where

$$c = \int_0^\infty\int_0^\infty \upsilon^{r-1}\,\omega^{\upsilon r-1}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\,e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\,d\omega$$

Hence, the marginal posterior distributions of υ and ω respectively, take the following forms:

$$\pi^*_1(\upsilon/\underline{x}) = \int_{\omega} \pi^*_{1,2}(\upsilon,\omega/\underline{x})\, d\omega = \int_0^{\infty} c^{-1}\upsilon^{r-1}\,\omega^{\upsilon r-1} \left(\prod_{i=1}^{r} x_i^{-\upsilon-1} e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\omega$$

and

$$\pi^*_2(\omega/\underline{x}) = \int_{\upsilon} \pi^*_{1,2}(\upsilon,\omega/\underline{x})\, d\upsilon = \int_0^{\infty} c^{-1}\upsilon^{r-1}\,\omega^{\upsilon r-1} \left(\prod_{i=1}^{r} x_i^{-\upsilon-1} e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon$$

The BI estimators for SE loss function (BISE) of υ and ω, denoted by, $\ddot{\upsilon}_{(SE)_{NIP}}$ and $\ddot{\omega}_{(SE)_{NIP}}$ were obtained as shown below:

$$\ddot{\upsilon}_{(SE)_{NIP}} = E(\upsilon/\underline{x}) = \int_{\upsilon} \upsilon\, \pi^*_1(\upsilon/\underline{x})\, d\upsilon$$

$$= c^{-1}\int_0^{\infty}\int_0^{\infty} \upsilon^{r}\,\omega^{\upsilon r-1} \left(\prod_{i=1}^{r} x_i^{-\upsilon-1} e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega, \quad (12)$$

and

$$\ddot{\omega}_{(SE)_{NIP}} = E(\upsilon/\underline{x}) = \int_{\omega} \omega\, \pi^*_2(\omega/\underline{x})\, d\omega$$

$$= c^{-1}\int_0^{\infty}\int_0^{\infty} \upsilon^{r-1}\,\omega^{\upsilon r} \left(\prod_{i=1}^{r} x_i^{-\upsilon-1} e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega, \quad (13)$$

Furthermore, the BI estimators for LINEX loss function (BILIN) of IW distribution unknown parameters, say, $\ddot{\upsilon}_{(LINEX)}$ and $\ddot{\omega}_{(LINEX)}$ respectively, were given as: For β is a real number and $\beta \neq 0$,

$$\ddot{\upsilon}_{(LINEX)_{NIP}} = \frac{-1}{\beta}\log E(e^{-\upsilon\beta}) = \frac{-1}{\beta}\log\left[\int_0^\infty e^{-\upsilon\beta}\pi*_1(\upsilon/\underline{x})\,d\upsilon\right],$$

$$= \frac{-1}{\beta}\log\left[c^{-1}\int_0^\infty\int_0^\infty \upsilon^{r-1}\omega^{\upsilon r-1}e^{-\upsilon\beta}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega,\right] \quad (14)$$

and,

$$\ddot{\omega}_{(LINEX)_{NIP}} = \frac{-1}{\beta}\log E(e^{-\omega\beta}) = \frac{-1}{\beta}\log\left[\int_0^\infty e^{-\omega\beta}\pi*_2(\omega/\underline{x})\,d\omega\right],$$

$$= \frac{-1}{\beta}\log\left[c^{-1}\int_0^\infty\int_0^\infty e^{-\omega\beta}\upsilon^{r-1}\omega^{\upsilon r-1}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega\,,\right] \quad (15)$$

The integral equations from (12) to (15) are very difficult to solve analytically. Therefore, the MCMC approach will be applied to approximate these integrals. The MH algorithm will be employed to calculate the BI estimators for different loss functions. Using equation (5), the BI estimators of $RT(\alpha,t)$, denoted by $\ddot{R}T_{(SE)_{NIP}}(\alpha,t)$ under the SE loss function and $\ddot{R}T_{(LINEX)_{NIP}}(\alpha,t)$ the LINEX loss function, respectively are given as follows

$$\ddot{R}T_{(SE)_{NIP}}(\alpha,t) = (\alpha-1)^{-1}\left[1-\left(1-e^{-\left(\frac{\ddot{\omega}}{t}\right)^{\ddot{\upsilon}_{(SE)_{NIP}}}}\right)^{-1}\phi\gamma\left(\alpha+\frac{\alpha}{\ddot{\upsilon}_{(SE)_{NIP}}}+\frac{1}{\ddot{\upsilon}_{(SE)_{NIP}}},\alpha\ddot{\omega}_{(SE)_{NIP}}{}^{\alpha}t^{-\ddot{\upsilon}_{(SE)_{NIP}}}\right)\right]$$

and

$$\ddot{R}T_{(LINEX)_{NIP}}(\alpha,t) = (\alpha-1)^{-1}\left[1-\left(1-e^{-\left(\frac{\ddot{\omega}}{t}\right)^{\ddot{\upsilon}_{(LINEX)_{NIP}}}}\right)^{-1}\phi\,\gamma\left(\alpha+\frac{\alpha}{\ddot{\upsilon}_{(LINEX)_{NIP}}}+\frac{1}{\ddot{\upsilon}_{(LINEX)_{NIP}}},\alpha\ddot{\omega}_{(LINEX)_{NIP}}{}^{\alpha}t^{-\ddot{\upsilon}_{(LINEX)_{NIP}}}\right)\right]$$

4.2 Residual entropy of $\mathbf{PCT_I}$ Estimators in the case of IP

In this sub-section, BI estimators of RT for $\mathrm{PCT_I}$ will be obtained under the assumption of independent gamma prior distributions with parameters $\upsilon = \mathrm{Gamma}(a_1,b_1)$ and $\omega = \mathrm{Gamma}(a_2,b_2)$, see Dey et al. (2016) of the IW unknown parameters distribution.

The joint prior distribution density function is:

$$\tau_{1,2}(\upsilon,\omega/\underline{x}) \propto \upsilon^{a_1-1}\omega^{a_2-1}e^{-b_1\upsilon-b_2\omega},$$

Where (a_J,b_J) , $J=1,2$ are known and negative.

The joint posterior density function parameters observed data $\underline{x}=(x_{(1)},x_{(2)},\ldots,x_{(s)})$ is written as

$$\tau*_{1,2}(\upsilon,\omega/\underline{x}) = \frac{L(\upsilon,\omega/x_1,x_2,...,x_n)\ \ \tau_{1,2}(\upsilon,\omega/\underline{x})}{\int_0^\infty\int_0^\infty L(\upsilon,\omega/x_1,x_2,...,x_n)\ \ \tau_{1,2}(\upsilon,\omega/\underline{x})\ d\upsilon\, d\omega}$$

$$= \frac{\upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j}}{\int_0^\infty\int_0^\infty \upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega}$$

$$\tau*_{1,2}(\upsilon,\omega/\underline{x}) = c_1^{-1}\upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j}$$

where

$$c_1 = \int_0^\infty\int_0^\infty \upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega$$

Hence, the marginal posterior distributions of υ and ω take the following forms

$$\tau*_1(\upsilon/\underline{x}) = \int_\omega \tau_{1,2}^{*}(\upsilon,\omega/\underline{x})\ d\omega = c_1^{-1}\int_0^\infty \upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\omega$$

and

$$\tau*_2(\omega/\underline{x}) = \int_\upsilon \tau^{*}_{1,2}(\upsilon,\omega/\underline{x})\ d\upsilon = c_1^{-1}\int_0^\infty \upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon$$

The BI$_{SE}$ of υ and ω denoted by $\ddot{\upsilon}_{(SE)_{IP}}$ and $\ddot{\omega}_{(SE)_{IP}}$ are obtained as follows

$$\ddot{\upsilon}_{(SE)_{IP}} = E(\upsilon/\underline{x}) = \int_\upsilon \upsilon\ \tau*_1(\upsilon/\underline{x})\ d\upsilon$$

$$= c_1^{-1}\int_0^\infty\int_0^\infty \upsilon^{a_1-r-1}\,\omega^{a_2-\upsilon r-2}\ e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega, \quad (16)$$

and

$$\ddot{\omega}_{(SE)_{IP}} = E(\omega/\underline{x}) = \int_\omega \omega\ \tau*_2(\omega/\underline{x})\ d\omega$$

$$= c_1^{-1}\int_0^\infty\int_0^\infty \upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-1}\, e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\, e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon\, d\omega, \tag{17}$$

The above BI estimators $\ddot{\upsilon}_{(SE)_{IP}}$ and $\ddot{\omega}_{(SE)_{IP}}$ can be numerically evaluated for given values of a_1, b_1, a_2, b_2, r, m and $\underline{x}$.

The BI_{LIN} of IW unknown parameters, namely $\ddot{\upsilon}_{(LINEX)_{IP}}$ and $\ddot{\omega}_{(LINEX)_{IP}}$ are obtained as

$$\ddot{\upsilon}_{(LINEX)_{NIP}} = \frac{-1}{\beta}\log E(e^{-\upsilon\beta}) = \frac{-1}{\beta}\log\left[\int_0^\infty e^{-\upsilon\beta}\pi *_1(\upsilon/\underline{x})\,d\upsilon\right], \beta\neq 0$$

$$= \frac{-1}{\beta}\log\left[\int_0^\infty\int_0^\infty c_1^{-1}\upsilon^{a_1-r-2}\,\omega^{a_2-\upsilon r-2}\,e^{-b_1\upsilon-b_2\omega}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\,e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon d\omega,\right] \tag{18}$$

and

$$\ddot{\omega}_{(LINEX)_{NIP}} = \frac{-1}{\beta}\log E(e^{-\omega\alpha}) = \frac{-1}{\alpha}\log\left[\int_0^\infty e^{-\omega\alpha}\pi *_2(\omega/\underline{x})\,d\omega\right], \beta\neq 0$$

$$= \frac{-1}{\beta}\log\left[\int_0^\infty\int_0^\infty c_1^{-1}\upsilon^{a_1}\,\omega^{\upsilon+b_1-1}\,e^{-(a_2\upsilon+b_2\omega+\omega\alpha)}\left(\prod_{i=1}^{r} x_i^{-\upsilon-1}\,e^{-\left(\frac{\omega}{x_i}\right)^{\upsilon}}\right)\prod_{j=1}^{m}\left[1-e^{-\left(\frac{\omega}{T_j}\right)^{\upsilon}}\right]^{R_j} d\upsilon d\omega\,,\right] \tag{19}$$

since β is a real number. The $\ddot{\upsilon}_{(LINEX)_{IP}}$ and $\ddot{\omega}_{(LINEX)_{IP}}$ estimators can be numerically calculated for given values of a_1, b_1, a_2, b_2, r, m and $\underline{x}$.

The equations from (16) to (19) are very hard to solve analytically. Therefore, the MCMC approach will be applied to approximate these integrals. The MH algorithm will be carried out to calculate the BI estimators for different loss functions. Using (5), the BI estimators of $RT(\alpha,t)$, denoted by $\ddot{R}T_{(SE)_{IP}}(\alpha,t)$ under SE loss function and $\ddot{R}T_{(LINEX)_{IP}}(\alpha,t)$ under LINEX loss function are given as follows

$$\ddot{R}T_{(SE)_{IP}}(\alpha,t) = (\alpha-1)^{-1}\left[1-\left(1-e^{-\left(\frac{\ddot{\omega}}{t}\right)^{\ddot{\upsilon}_{(SE)_{IP}}}}\right)^{-1}\phi\,\gamma\left(\alpha+\frac{\alpha}{\ddot{\upsilon}_{(SE)_{IP}}}+\frac{1}{\ddot{\upsilon}_{(SE)_{IP}}}, \alpha\ddot{\omega}_{(SE)_{IP}}{}^{\alpha}t^{-\ddot{\upsilon}_{(SE)_{IP}}}\right)\right]$$

and

$$\ddot{R}T_{(LINEX)_{IP}}(\alpha,t) = (\alpha-1)^{-1}\left[1-\left(1-e^{-\left(\frac{\ddot{\omega}}{t}\right)^{\ddot{\upsilon}_{(LINEX)_{IP}}}}\right)^{-1}\phi\,\gamma\left(\alpha+\frac{\alpha}{\ddot{\upsilon}_{(LINEX)_{IP}}}+\frac{1}{\ddot{\upsilon}_{(LINEX)_{IP}}}, \alpha\ddot{\omega}_{(LINEX)_{IP}}{}^{\alpha}t^{-\ddot{\upsilon}_{(LINEX)_{IP}}}\right)\right]$$

5. Application and Simulation Study

The purpose of this part is to test the performance of the suggested RT entropy estimators for the IW model using various estimation techniques discussed in the previous sections. To illustrate the theoretical results, we examine real data. Moreover, we proceed with a simulation study to assess the performance of the suggested estimation models and to evaluate the estimator's statistical performance in the case of $\mathrm{PCT_I}$ Scheme. For calculations, R statistical programming language is used. Further, we calculated ML estimators in R-statistical language by using the bbmle package.

5.1 Application

An application to actual real-world data is examined to demonstrate objectives and to evaluate the ML and BI estimators for the RT entropy of the IW distribution in the case of $\mathrm{PCT_I}$ scheme. According to Bhaumik et al. (2009), the data comprise 34 observations and were analyzed by testing the parameters of the Gamma distribution. The data represent vinyl chloride concentrations (mg/l) obtained from clean-up gradient monitoring wells. The data are listed as follows:

5.1	**1.2**	**1.3**	**0.6**	**0.5**	**2.4**	**0.5**	**1.1**	**8**	**0.8**	**0.4**	**0.6**	**0.9**	**0.4**	**2**	**0.5**	**5.3**
3.2	**2.7**	**2.9**	**2.5**	**2.3**	**1**	**0.2**	**0.1**	**0.1**	**1.8**	**0.9**	**2**	**4**	**6.8**	**1.2**	**0.4**	**0.2**

The IW distribution appears to be appropriate for fitting the data. This conclusion is supported by the estimated Kolmogorov-Smirnov distance (KS) test for the IW distribution between the empirical and fitted distribution, which 0.1134, with a corresponding p-value is 0.7749 where the ML estimators of $\ddot{\upsilon}=0.8803$, and $\ddot{\omega}=0.6174$. The empirical PDF and the empirical CDF of the IW distribution plots are shown in Figure 2, This figure points to the IW distribution is a good fit for the current data set.

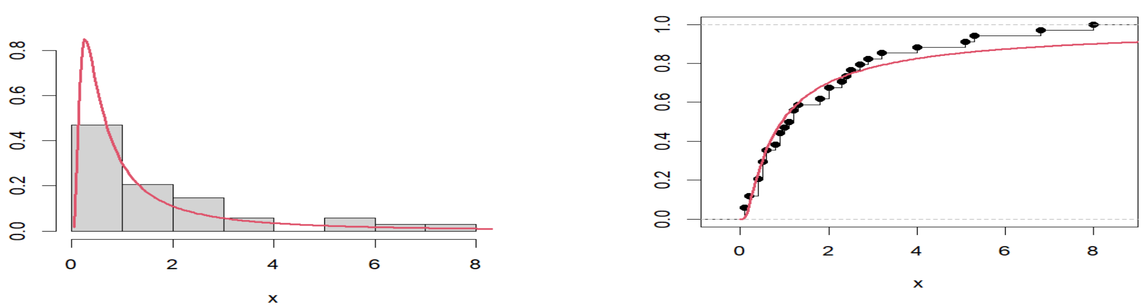

Figure 2. Empirical PDF and CDF IW distribution for real data

One can first check whether the IW distribution is suitable for analyzing this data set. The ML estimators of the parameters are reported, along with the values of the log-likelihood criterion (LLC), Akaike information criterion (AIC), Bayesian information criterion (BIC), and KS test statistic to assess the goodness of fit. These criteria are computed with IW, Inverse gamma (IG), generalized exponential (Gen-Exp) and Inverse exponential (Inv-Exp). A lower value of these criteria indicates a better fit. The parameter estimates and goodness-of-fit statistics are presented in Table (2).

From the original data, we generate four $\mathrm{PCT_I}$ schemes, namely, Scheme 1 ($\Im_1$), Scheme 2 ($\Im_2$), Scheme 3 ($\Im_3$), and Scheme 4 ($\Im_4$) with the number of stages $m = 3$ and incorporates the removal of items $\mathrm{R}_j, j = 1,\ldots,m$ with time censoring T. These different schemes can be described as follows:

$\Im_1 : R_j = (0,0,R_m)$

$\Im_2 : R_j = (2,2,R_m)$

$\Im_3 : R_j = (3,3,R_m)$

$\Im_4 : R_j = (4,0,R_m)$

Two separate values of α and constant t are used, respectively (1.5, 2.5) and (0.5, 1.5). Note that: Sch.1 represents the usual $\mathrm{PCT_I}$ scheme. Table 1 presents the ML and BI estimators of RT entropy at different $\mathrm{PCT_I}$ schemes under SE, and LINEX loss functions for the IW distribution. Moreover, BI estimates can be evaluated using the MH algorithm under the NIP function for different loss functions SE, LIN-I ($\beta = .05$), and LIN-II ($\beta = -.05$) are utilized.

Table 1. ML and BI (NIP of RT (α, t)) estimators for the given real data under different $\mathrm{PCT_I}$ schemes from IW distribution with $m = 3$ and $\mathrm{T} = (0.50, 1.20, 2.50)$.

α	t	schemes	ML	BI$_{SE}$	BI$_{LIN\text{-}I}$	BI$_{LIN\text{-}II}$
1.5	**0.5**	$\Im_1$	1.942503	1.948480	1.946530	1.950437
		$\Im_2$	1.934531	1.951314	1.949462	1.953225
		$\Im_3$	1.970604	1.978560	1.978300	1.978601
		$\Im_4$	1.935500	1.951700	1.949430	1.953971
	1.5	$\Im_1$	1.9619851	1.968477	1.967655	1.969266
		$\Im_2$	1.940757	1.963980	1.961677	1.966476
		$\Im_3$	1.965348	1.983178	1.982726	1.983316
		$\Im_4$	1.955062	1.963604	1.962287	1.964916
2.5	**0.5**	$\Im_1$	0.666645	0.666654	0.666654	0.666654
		$\Im_2$	0.666653	0.666653	0.666653	0.666653
		$\Im_3$	0.666655	0.666658	0.666657	0.666658
		$\Im_4$	0.666646	0.666656	0.666655	0.666657
	1.5	$\Im_1$	0.666662	0.666664	0.666663	0.666664
		$\Im_2$	0.666665	0.666666	0.666666	0.666666
		$\Im_3$	0.666663	0.666664	0.666664	0.666664
		$\Im_4$	0.666664	0.666665	0.666665	0.666665

Regarding the tabulated values in Table 1, $\mathfrak{I}_3$ is the best for estimating RT(α,t). also, the estimates of RT(α,t) increases as t increases.

Table 2. Goodness-of-fit tests for survival times.

Distribution	ML		LLC	AIC	BIC	HQIC	K-S	P-value
IW	0.88031	0.61740	58.62659	121.25317	124.30589	122.29424	0.11335	0.77489
IG	0.90016	0.51535	59.06594	122.13187	125.18459	123.17294	0.13101	0.60380
Gen-Exp	0.92130	0.54121	59.12846	122.25692	125.30964	123.29799	0.13657	0.55012
Inv-Exp	0.57266	----	59.19303	122.38605	125.91241	123.90658	0.14690	0.45538
gamma	1.06246	1.76867	59.41316	124.82633	125.87905	125.86739	0.19602	0.14659

Also, the histogram can be plotted alongside corresponding fitted PDF lines for the same distributions. Figure 3 illustrates the fitted lines for both CDF and PDF for the given dataset and the corresponding distributions. The figures also indicate that the IW distribution provides a better fit compared to the other distributions, at least for this dataset.

Emiprical and theortical CDF **Histogram**

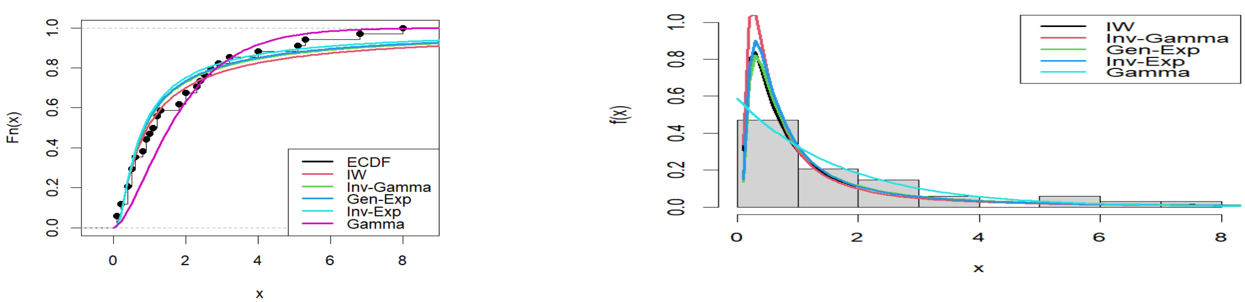

Figure 3. Estimated PDF and CDF with corresponding distributions for the data set

5.2 Simulation study

For IW distribution under the $\mathrm{PCT_I}$ scheme, we utilize a Monte Carlo simulation to evaluate the performance of different estimation methods, particularly ML and BI, which are employed in this work. We generate 1,000 ML estimates of the IW model under the following assumptions:

1. Assuming parameters of IW model (ν,ω) as $(1.5, 2.5)$ and $(2.5, 1.5)$.
2. The value of $\alpha = 1.5,2.5$ and the constant value of is $t = (0.5,1.5)$.
3. The true value of $\mathrm{RT}(\alpha,t)$, denoted by RT has the following values:

$IW(\nu,\omega)$	α	t	RT(α, t)
$IW(1.5, 2.5)$	1.5	0.5	1.90599
		1.5	1.90848
	2.5	0.5	0.66630
		1.5	0.66632
$IW(2.5, 1.5)$	1.5	0.5	1.86654
		1.5	1.88596
	2.5	0.5	0.66487
		1.5	0.66544

4. The sample sizes are assumed to be $n = 40, 60, 120$.
5. The number of stages of the $\mathrm{PCT_I}$ scheme is $m = 3$ and 5 and vector of different time censoring T is:

$IW(\nu,\omega)$	m	$T = (T_1, T_2, \dots, T_m)$
$IW(1.5, 2.5)$	3	$T = (2.00, 3.20, 5.80)$
	5	$T = (1.45, 2.00, 2.65.3.50, 5.00)$
$IW(2.5, 1.5)$	3	$T = (1.30,1.75,2.50)$
	5	$T = (1.00,1.35, 1.55,1.85,2.30)$

6. Removed items R_j are assumed at n and m, where $R_m = n - (\sum_{j=1}^{m-1} R_j + r)$, r is the number of failure items with the following censoring schemes

$m = 3$			$m = 5$		
n	Removing Items $R = (R_1, R_2, R_3)$	Index Scheme	n	Removing Items $R = (R_1, R_2, R_3, R_4, R_5)$	Index Scheme
40	$R = (0,0, R_m)$	Sch.1	40	$R = (0,0,0,0, R_m)$	Sch.1
	$R = (3,3, R_m)$	Sch.2		$R = (2,2,2,2, R_m)$	Sch.2
	$R = (6,6, R_m)$	Sch.3		$R = (4,4,4,4, R_m)$	Sch.3
	$R = (12,0, R_m)$	Sch.4		$R = (8,0,0,0, R_m)$	Sch.4
60	$R = (0,0, R_m)$	Sch.1	60	$R = (0,0,0,0, R_m)$	Sch.1
	$R = (4,4, R_m)$	Sch.2		$R = (3,3,3,3, R_m)$	Sch.2
	$R = (8,8, R_m)$	Sch.3		$R = (5,5,5,5, R_m)$	Sch.3
	$R = (8,0, R_m)$	Sch.4		$R = (12,0,0,0, R_m)$	Sch.4
120	$R = (0,0, R_m)$	Sch.1	120	$R = (0,0,0,0, R_m)$	Sch.1
	$R = (8,8, R_m)$	Sch.2		$R = (5,5,5,5, R_m)$	Sch.2
	$R = (16,16, R_m)$	Sch.3		$R = (10,10,10,10, R_m)$	Sch.3
	$R = (16,0, R_m)$	Sch.4		$R = (20,0,0,0, R_m)$	Sch.4

ML estimators are evaluated $\mathrm{PCT_I}$ based on the previous assumptions and the generated data. These ML estimator values are then utilized to calculate the RT entropy given α and t. In the computation of the BI method, we employ the MH algorithm under different loss functions SE and LINEX in the case of NIP and IP are discussed, where the hyper parameters in IP function are assumed as follows :

- Case I: $\nu = 1.5$ and $\omega = 2.5$

$$a_1 = 90.62, \quad b_1 = 58.90, \; a_2 = 120.37, \;\; b_2 = 47.92$$

- Case II: $\nu = 2.5$ and $\omega = 1.5$

$$a_1 = 93.61, \quad b_1 = 36.60, \; a_2 = 359.44, \;\; b_2 = 238.60$$

Such values of informative priors are plugged in to determine the required estimates. Through implementation of the MH algorithm, the ML estimators are used as initial guess values, as well as the corresponding variance-covariance matrix of $(\ln(\ddot{\alpha}),\ \ln_{(\ddot{\omega})})$. Different loss functions, including SE, LN-I ($\beta = 0.5$) and LN-II ($\beta = -0.5$) loss functions are used. These values are then utilized to find the estimated values. Finally, 2,000 burn-in samples are removed from the total 10,000 samples created by the posterior density, and the estimates of $\mathrm{RT}(\alpha, t)$ are derived. The Bias (BIA) and Root Mean Square Error (RMSE) for all RT entropy estimates are listed in Tables (3-12) covering all inputs of the Monto Carlo simulation. From the tabulated values, we will conclude the following remarks:

1. In general, the RMSE values tend to decrease as the sample size increases.
2. With the growth of time t, the Bia and RMSE values of estimators decrease.
3. RMSE of BI estimates under the IP gradually decreases as n and m increase.
4. RT entropy estimators utilizing the IP loss function generally exhibit significantly better performance compared to those employing the NIP loss function in most cases.
5. Under both BIA and RMSE criteria, BI estimators under the LIN-II loss function typically outperform other competing BI estimators.
6. When α and t increasing, the BI of RT (α, t) under all methods will decrease, as shown in Tables 11 and 12.
7. BI estimators utilizing the IP loss function perform better than ML estimators, while ML estimators compete well than BI estimators using the NIP loss function.
8. The BI estimators of RT (α, t) decrease with the growth of the sample size.

Table 3. Estimates of BIA and RMSE of RT(1.5,0.5) for IW ($\nu = 1.5, \omega = 2.5$) under PCT_1 with different censoring schemes and $m = 3$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.013653	0.047053	0.055533	0.320360	0.528549	10.176981	0.037402	0.107611	0.003419	0.010910	0.003481	0.010976	0.003357	0.010845
	Sch.2	0.008577	0.033831	0.040042	0.139634	0.081483	0.510737	0.033673	0.104646	0.003219	0.009080	0.003277	0.009134	0.003161	0.009027
	Sch.3	0.013488	0.075111	0.066252	0.494903	0.742569	11.142358	0.035628	0.131197	0.002244	0.009110	0.002303	0.009168	0.002186	0.009054
	Sch.4	0.018569	0.061993	0.121332	1.117469	1.031785	15.557343	0.053857	0.192573	0.003519	0.010758	0.003582	0.010825	0.003456	0.010692
60	Sch.1	0.006692	0.029398	0.018201	0.058114	0.019891	0.072306	0.017172	0.052634	0.002722	0.010202	0.002772	0.010254	0.002671	0.010150
	Sch.2	0.005245	0.028170	0.029338	0.232178	0.286931	5.916228	0.020150	0.079686	0.002268	0.009246	0.002317	0.009294	0.002219	0.009198
	Sch.3	0.007491	0.036984	0.030598	0.118967	0.083024	0.840192	0.025361	0.080627	0.002304	0.009427	0.002354	0.009473	0.002254	0.009381
	Sch.4	0.005956	0.029522	0.025056	0.090753	0.032078	0.190984	0.022893	0.075213	0.002497	0.009239	0.002547	0.009287	0.002448	0.009191
120	Sch.1	0.002661	0.016171	0.006849	0.021377	0.006971	0.021595	0.006730	0.021166	0.001800	0.008785	0.001832	0.008813	0.001769	0.008758
	Sch.2	0.004929	0.019962	0.009834	0.026773	0.010053	0.027349	0.009631	0.026276	0.003037	0.010452	0.003073	0.010490	0.003000	0.010415
	Sch.3	0.003686	0.016837	0.009423	0.027031	0.009961	0.032665	0.009147	0.025383	0.002609	0.008678	0.002646	0.008712	0.002572	0.008645
	Sch.4	0.002833	0.017670	0.008393	0.025271	0.008567	0.025638	0.008225	0.024926	0.001683	0.009017	0.001717	0.009046	0.001648	0.008988

Table 4. Estimates of BIA and RMSE of RT(1.5,0.5) for IW ($\nu = 1.5, \omega = 2.5$) under PCT_1 with different censoring schemes and $m = 5$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.009515	0.036211	0.139686	2.502119	0.036836	0.200201	0.027085	0.081244	0.003644	0.011610	0.003717	0.011691	0.003571	0.011531
	Sch.2	0.016260	0.086653	0.634941	1.824642	0.331760	3.019394	0.044992	0.157103	0.003436	0.011901	0.003513	0.011999	0.003359	0.011805
	Sch.3	0.020409	0.091802	5.341757	2.252047	0.470749	3.761831	0.071758	0.285176	0.004149	0.011922	0.004230	0.012015	0.004069	0.011830
	Sch.4	0.017352	0.075076	0.170877	1.428330	2.147518	2.586061	0.062314	0.235227	0.003617	0.011872	0.003699	0.011964	0.003537	0.011782
60	Sch.1	0.007529	0.029368	0.018974	0.052471	0.020293	0.057943	0.017958	0.049032	0.003349	0.011310	0.003412	0.011379	0.003286	0.011241
	Sch.2	0.004898	0.030565	0.073193	1.212019	1.295204	2.495337	0.023388	0.138175	0.002419	0.010729	0.002479	0.010791	0.002360	0.010667
	Sch.3	0.009188	0.037506	0.029511	0.103303	0.037307	0.169511	0.026454	0.086449	0.003192	0.011151	0.003255	0.011220	0.003130	0.011082
	Sch.4	0.009197	0.050803	0.027128	0.082075	0.031580	0.113387	0.024812	0.069933	0.003189	0.010949	0.003253	0.011021	0.003127	0.010879
120	Sch.1	0.003217	0.018716	0.008380	0.026168	0.008550	0.026575	0.008218	0.025788	0.002369	0.010472	0.002407	0.010510	0.002332	0.010434
	Sch.2	0.003537	0.017194	0.009353	0.025838	0.009523	0.026195	0.009189	0.025500	0.003133	0.010602	0.003175	0.010646	0.003091	0.010559
	Sch.3	0.001969	0.019737	0.008139	0.028990	0.008409	0.030333	0.007894	0.027912	0.001664	0.009701	0.001705	0.009739	0.001623	0.009663
	Sch.4	0.004073	0.019965	0.009851	0.027846	0.010101	0.028635	0.009622	0.027183	0.002851	0.010719	0.002893	0.010763	0.002809	0.010676

Table 5. Estimates of BIA and RMSE of RT(2.5,1.5) for IW($\nu = 1.5, \omega = 2.5$) under PCT_1 with different censoring schemes and $m = 3$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.000189	0.001062	0.001147	0.008643	0.001370	0.011167	0.001023	0.007324	0.000042	0.000121	0.000042	0.000121	0.000042	0.000121
	Sch.2	0.000139	0.000427	0.002159	0.037613	0.046839	1.036203	0.000934	0.010630	0.000047	0.000112	0.000047	0.000112	0.000047	0.000112
	Sch.3	0.000178	0.000914	0.010358	0.111022	0.330284	4.586602	0.003911	0.035484	0.000037	0.000110	0.000037	0.000110	0.000037	0.000110
	Sch.4	0.000143	0.000468	0.003378	0.060636	0.067976	1.504249	0.001596	0.021548	0.000039	0.000107	0.000039	0.000107	0.000039	0.000107
60	Sch.1	0.000068	0.000220	0.000310	0.003590	0.000370	0.004901	0.000275	0.002840	0.000034	0.000097	0.000034	0.000097	0.000034	0.000097
	Sch.2	0.000059	0.000247	2.695409	1.913850	0.000115	0.000426	0.002836	0.060837	0.000028	0.000091	0.000028	0.000091	0.000028	0.000091
	Sch.3	0.000053	0.000173	0.000145	0.000569	0.000145	0.000571	0.000145	0.000567	0.000032	0.000094	0.000032	0.000094	0.000032	0.000094
	Sch.4	0.000073	0.000210	0.008155	0.146456	0.945610	2.325999	0.001322	0.018520	0.000039	0.000097	0.000039	0.000097	0.000039	0.000097
120	Sch.1	0.000029	0.000115	0.000048	0.000177	0.000048	0.000177	0.000048	0.000177	0.000022	0.000076	0.000022	0.000076	0.000022	0.000076
	Sch.2	0.000030	0.000106	0.000046	0.000144	0.000046	0.000144	0.000046	0.000144	0.000024	0.000075	0.000024	0.000075	0.000024	0.000075
	Sch.3	0.000032	0.000102	0.000052	0.000138	0.000052	0.000138	0.000052	0.000138	0.000028	0.000077	0.000028	0.000077	0.000028	0.000077
	Sch.4	0.000024	0.000100	0.000043	0.000145	0.000043	0.000145	0.000043	0.000145	0.000020	0.000073	0.000020	0.000073	0.000020	0.000073

Table 6: Estimates of BIA and RMSE of RT(2.5,1.5) or IW($\nu = 1.5, \omega = 2.5$) under PCT_1 with different censoring schemes and $m = 5$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.000122	0.000401	0.000698	0.004325	0.000786	0.005593	0.000649	0.003743	0.000039	0.000102	0.000039	0.000102	0.000039	0.000102
	Sch.2	0.000176	0.000913	2.819203	2.168650	0.030104	0.650171	0.007558	0.136613	0.000031	0.000092	0.000031	0.000092	0.000031	0.000092
	Sch.3	0.000295	0.001920	0.015945	0.202427	0.722883	1.981696	0.003864	0.032024	0.000044	0.000105	0.000044	0.000105	0.000044	0.000105
	Sch.4	0.000239	0.001010	0.004902	0.053309	0.184595	2.814315	0.002452	0.022233	0.000050	0.000116	0.000050	0.000116	0.000050	0.000116
60	Sch.1	0.000062	0.000210	0.000129	0.000504	0.000129	0.000506	0.000129	0.000503	0.000026	0.000086	0.000026	0.000086	0.000026	0.000086
	Sch.2	0.000084	0.000257	0.000527	0.004616	0.000622	0.005941	0.000466	0.003792	0.000039	0.000100	0.000039	0.000100	0.000039	0.000100
	Sch.3	0.000075	0.000233	0.004302	0.085935	0.655224	1.636415	0.000736	0.007888	0.000030	0.000086	0.000030	0.000086	0.000030	0.000086
	Sch.4	0.000096	0.000349	0.003537	0.069789	0.094766	2.109202	0.000846	0.010301	0.000035	0.000096	0.000035	0.000096	0.000035	0.000096
120	Sch.1	0.000035	0.000140	0.000050	0.000171	0.000050	0.000171	0.000050	0.000171	0.000024	0.000083	0.000024	0.000083	0.000024	0.000083
	Sch.2	0.000029	0.000117	0.000050	0.000187	0.000050	0.000188	0.000050	0.000187	0.000022	0.000074	0.000022	0.000074	0.000022	0.000074
	Sch.3	0.000030	0.000096	0.000057	0.000153	0.000057	0.000153	0.000057	0.000153	0.000021	0.000066	0.000021	0.000066	0.000021	0.000066
	Sch.4	0.000034	0.000135	0.000061	0.000227	0.000061	0.000228	0.000061	0.000227	0.000025	0.000081	0.000025	0.000081	0.000025	0.000081

Table 7. Estimates of BIA and RMSE of RT(1.5,0.5) for IW($\nu = 2.5, \omega = 1.5$) under PCT$_1$ with different censoring schemes and $m = 3$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.001640	0.035320	0.007855	0.038356	0.008222	0.038747	0.007495	0.037980	0.000928	0.013682	0.001030	0.013721	0.000826	0.013643
	Sch.2	0.000448	0.036356	0.005497	0.039774	0.005847	0.040160	0.005154	0.039402	0.000162	0.013491	0.000265	0.013521	0.000060	0.013461
	Sch.3	0.002347	0.037809	0.007718	0.043267	0.008094	0.043668	0.007348	0.042879	0.000633	0.013463	0.000738	0.013502	0.000528	0.013426
	Sch.4	0.006784	0.042740	0.011865	0.048354	0.012305	0.048916	0.011435	0.047814	0.001866	0.014390	0.001974	0.014437	0.001760	0.014344
60	Sch.1	0.002089	0.030543	0.005707	0.031874	0.005922	0.032069	0.005494	0.031683	0.001104	0.014268	0.001192	0.014304	0.001016	0.014234
	Sch.2	0.002647	0.028527	0.005547	0.031347	0.005775	0.031541	0.005322	0.031157	0.001790	0.013778	0.001879	0.013817	0.001701	0.013740
	Sch.3	0.003407	0.030089	0.006644	0.033139	0.006882	0.033366	0.006408	0.032917	0.001740	0.013462	0.001829	0.013500	0.001652	0.013426
	Sch.4	0.001105	0.028448	0.003944	0.030951	0.004167	0.031132	0.003724	0.030775	0.000521	0.012855	0.000608	0.012883	0.000434	0.012828
120	Sch.1	0.000528	0.019940	0.002657	0.021084	0.002753	0.021144	0.002562	0.021026	0.000482	0.012954	0.000539	0.012973	0.000425	0.012936
	Sch.2	0.001507	0.019664	0.000265	0.020140	0.000172	0.020183	0.000358	0.020098	0.000742	0.012561	0.000685	0.012574	0.000798	0.012548
	Sch.3	0.001546	0.020741	0.003113	0.021918	0.003216	0.021982	0.003010	0.021856	0.001128	0.012889	0.001188	0.012913	0.001068	0.012865
	Sch.4	0.000509	0.020421	0.002181	0.021244	0.002285	0.021306	0.002077	0.021183	0.000463	0.012634	0.000522	0.012654	0.000404	0.012614

Table 8. Estimates of BIA and RMSE of RT(1.5,0.5) for IW($\nu = 2.5, \omega = 1.5$) under PCT1 with different censoring schemes and $m = 5$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.002005	0.035450	0.008477	0.040703	0.008858	0.041154	0.008104	0.040270	0.003131	0.015498	0.003250	0.015560	0.003013	0.015438
	Sch.2	0.002910	0.038554	0.007584	0.041054	0.007968	0.041468	0.007209	0.040656	0.002943	0.016298	0.003068	0.016364	0.002819	0.016234
	Sch.3	0.004023	0.041087	0.008028	0.045145	0.008436	0.045594	0.007628	0.044711	0.003458	0.016202	0.003590	0.016277	0.003328	0.016129
	Sch.4	0.002846	0.035577	0.008747	0.041448	0.009163	0.041901	0.008341	0.041012	0.002974	0.014292	0.003102	0.014359	0.002847	0.014227
60	Sch.1	0.003169	0.029024	0.006859	0.031474	0.007087	0.031673	0.006634	0.031278	0.003492	0.015621	0.003591	0.015674	0.003393	0.015569
	Sch.2	0.000446	0.029766	0.004578	0.032770	0.004806	0.032979	0.004354	0.032565	0.002194	0.015226	0.002294	0.015279	0.002095	0.015175
	Sch.3	0.002539	0.030816	0.005211	0.032625	0.005451	0.032832	0.004973	0.032423	0.002857	0.015712	0.002961	0.015768	0.002754	0.015657
	Sch.4	0.002438	0.030073	0.006001	0.033254	0.006250	0.033470	0.005756	0.033044	0.002571	0.014736	0.002675	0.014787	0.002467	0.014687
120	Sch.1	0.002181	0.019391	0.003757	0.020236	0.003854	0.020296	0.003660	0.020177	0.002837	0.013920	0.002902	0.013954	0.002773	0.013886
	Sch.2	0.000610	0.019771	0.002141	0.020258	0.002241	0.020313	0.002042	0.020204	0.001498	0.013457	0.001562	0.013484	0.001435	0.013430
	Sch.3	0.001679	0.020969	0.003070	0.021985	0.003181	0.022053	0.002959	0.021918	0.002381	0.013816	0.002449	0.013850	0.002313	0.013784
	Sch.4	0.000023	0.020618	0.001252	0.021275	0.001357	0.021331	0.001148	0.021220	0.001315	0.013896	0.001380	0.013923	0.001249	0.013869

Table 9. Estimates of BIA and RMSE of RT(2.5,0.5) or IW($\nu = 2.5, \omega = 1.5$) under PCT1 with different censoring schemes and $m = 3$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.000296	0.001263	0.000477	0.001544	0.000478	0.001545	0.000476	0.001543	0.000137	0.000425	0.000137	0.000426	0.000137	0.000425
	Sch.2	0.000384	0.001417	0.000606	0.001818	0.000606	0.001820	0.000605	0.001815	0.000136	0.000416	0.000136	0.000416	0.000136	0.000415
	Sch.3	0.000425	0.001661	0.000544	0.001924	0.000545	0.001926	0.000543	0.001922	0.000139	0.000439	0.000139	0.000439	0.000139	0.000438
	Sch.4	0.000490	0.001768	0.000755	0.002328	0.000756	0.002331	0.000754	0.002326	0.000170	0.000470	0.000170	0.000470	0.000170	0.000470
60	Sch.1	0.000298	0.001181	0.000416	0.001338	0.000416	0.001339	0.000415	0.001337	0.000164	0.000488	0.000164	0.000488	0.000164	0.000488
	Sch.2	0.000241	0.001059	0.000364	0.001255	0.000364	0.001255	0.000364	0.001254	0.000136	0.000433	0.000136	0.000434	0.000136	0.000433
	Sch.3	0.000184	0.001079	0.000250	0.001191	0.000251	0.001192	0.000250	0.001190	0.000110	0.000428	0.000110	0.000428	0.000110	0.000428
	Sch.4	0.000279	0.001166	0.000368	0.001442	0.000368	0.001443	0.000367	0.001442	0.000141	0.000432	0.000141	0.000432	0.000141	0.000432
120	Sch.1	0.000117	0.000644	0.000162	0.000693	0.000163	0.000693	0.000162	0.000693	0.000110	0.000392	0.000110	0.000392	0.000109	0.000392
	Sch.2	0.000136	0.000702	0.000181	0.000763	0.000181	0.000763	0.000181	0.000763	0.000114	0.000411	0.000114	0.000411	0.000114	0.000411
	Sch.3	0.000133	0.000684	0.000178	0.000789	0.000178	0.000789	0.000178	0.000789	0.000109	0.000397	0.000109	0.000397	0.000109	0.000397
	Sch.4	0.000168	0.000706	0.000205	0.000789	0.000205	0.000789	0.000205	0.000789	0.000123	0.000393	0.000123	0.000393	0.000123	0.000393

Table 10. Estimates of BIA and RMSE of RT(2.5,0.5) for IW($\nu = 2.5, \omega = 1.5$) under PCT_1 with different censoring schemes and $m = 5$

n	Sch.	MLE		BI-NIP						BI-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.000240	0.001199	0.000407	0.001526	0.000407	0.001527	0.000406	0.001525	0.000084	0.000365	0.000084	0.000365	0.000084	0.000365
	Sch.2	0.000369	0.001392	0.000555	0.001849	0.000556	0.001851	0.000554	0.001846	0.000117	0.000380	0.000117	0.000381	0.000117	0.000380
	Sch.3	0.000612	0.001816	0.000808	0.002316	0.000809	0.002319	0.000807	0.002313	0.000157	0.000400	0.000157	0.000400	0.000157	0.000400
	Sch.4	0.000444	0.001588	0.000641	0.002010	0.000642	0.002012	0.000640	0.002008	0.000130	0.000393	0.000130	0.000393	0.000130	0.000393
60	Sch.1	0.000217	0.000983	0.000346	0.001192	0.000347	0.001193	0.000346	0.001192	0.000111	0.000398	0.000112	0.000398	0.000111	0.000398
	Sch.2	0.000293	0.001062	0.000397	0.001326	0.000398	0.001326	0.000397	0.001325	0.000151	0.000404	0.000151	0.000404	0.000151	0.000404
	Sch.3	0.000344	0.001249	0.000438	0.001415	0.000438	0.001415	0.000437	0.001414	0.000135	0.000425	0.000135	0.000425	0.000135	0.000425
	Sch.4	0.000309	0.001293	0.000430	0.001513	0.000430	0.001514	0.000430	0.001512	0.000119	0.000418	0.000119	0.000418	0.000119	0.000418
120	Sch.1	0.000122	0.000666	0.000163	0.000738	0.000163	0.000739	0.000163	0.000738	0.000100	0.000391	0.000101	0.000391	0.000100	0.000391
	Sch.2	0.000142	0.000737	0.000187	0.000783	0.000188	0.000783	0.000187	0.000782	0.000104	0.000385	0.000104	0.000385	0.000104	0.000385
	Sch.3	0.000188	0.000771	0.000196	0.000798	0.000196	0.000798	0.000196	0.000798	0.000122	0.000402	0.000122	0.000402	0.000122	0.000402
	Sch.4	0.000125	0.000700	0.000176	0.000778	0.000176	0.000779	0.000176	0.000778	0.000096	0.000380	0.000096	0.000380	0.000096	0.000380

Table 11. Estimates of BIA and RMSE of RT(1.5,0.5) for IW ($\nu = 2.5, \omega = 1.5$) under PCT$_1$ with different censoring schemes and $m = 3$

n	Sch.	MLE		BE-NIP						BE-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.001640	0.035320	0.007855	0.038356	0.008222	0.038747	0.007495	0.037980	0.000928	0.013682	0.001030	0.013721	0.000826	0.013643
	Sch.2	0.000448	0.036356	0.005497	0.039774	0.005847	0.040160	0.005154	0.039402	0.000162	0.013491	0.000265	0.013521	0.000060	0.013461
	Sch.3	0.002347	0.037809	0.007718	0.043267	0.008094	0.043668	0.007348	0.042879	0.000633	0.013463	0.000738	0.013502	0.000528	0.013426
	Sch.4	0.006784	0.042740	0.011865	0.048354	0.012305	0.048916	0.011435	0.047814	0.001866	0.014390	0.001974	0.014437	0.001760	0.014344
60	Sch.1	0.002089	0.030543	0.005707	0.031874	0.005922	0.032069	0.005494	0.031683	0.001104	0.014268	0.001192	0.014304	0.001016	0.014234
	Sch.2	0.002647	0.028527	0.005547	0.031347	0.005775	0.031541	0.005322	0.031157	0.001790	0.013778	0.001879	0.013817	0.001701	0.013740
	Sch.3	0.003407	0.030089	0.006644	0.033139	0.006882	0.033366	0.006408	0.032917	0.001740	0.013462	0.001829	0.013500	0.001652	0.013426
	Sch.4	0.001105	0.028448	0.003944	0.030951	0.004167	0.031132	0.003724	0.030775	0.000521	0.012855	0.000608	0.012883	0.000434	0.012828
120	Sch.1	0.000528	0.019940	0.002657	0.021084	0.002753	0.021144	0.002562	0.021026	0.000482	0.012954	0.000539	0.012973	0.000425	0.012936
	Sch.2	0.001507	0.019664	0.000265	0.020140	0.000172	0.020183	0.000358	0.020098	0.000742	0.012561	0.000685	0.012574	0.000798	0.012548
	Sch.3	0.001546	0.020741	0.003113	0.021918	0.003216	0.021982	0.003010	0.021856	0.001128	0.012889	0.001188	0.012913	0.001068	0.012865
	Sch.4	0.000509	0.020421	0.002181	0.021244	0.002285	0.021306	0.002077	0.021183	0.000463	0.012634	0.000522	0.012654	0.000404	0.012614

Table 12. Estimates of BIA and RMSE of RT(1.5,1.5) for IW($\nu = 2.5, \omega = 1.5$) under PCT1 with different censoring schemes and $m = 3$

n	Sch.	MLE		BE-NIP						BE-IP					
				SE		LN-I		LN-II		SE		LN-I		LN-II	
		BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE	BIA	RMSE
40	Sch.1	0.000188	0.022400	0.002862	0.024546	0.003000	0.024652	0.002725	0.024443	0.000500	0.008927	0.000547	0.008942	0.000454	0.008913
	Sch.2	0.000181	0.022163	0.001885	0.024082	0.002020	0.024175	0.001751	0.023992	0.000655	0.008886	0.000702	0.008901	0.000609	0.008871
	Sch.3	0.001441	0.026176	0.002671	0.027993	0.002816	0.028113	0.002527	0.027874	0.000776	0.009756	0.000824	0.009772	0.000728	0.009739
	Sch.4	0.000531	0.024968	0.000688	0.025781	0.000554	0.025873	0.000820	0.025690	0.000134	0.009118	0.000182	0.009132	0.000087	0.009105
60	Sch.1	0.000394	0.018320	0.002341	0.019672	0.002432	0.019731	0.002250	0.019614	0.000355	0.009127	0.000393	0.009138	0.000316	0.009116
	Sch.2	0.000818	0.018925	0.000330	0.019784	0.000246	0.019832	0.000414	0.019737	0.000139	0.009225	0.000101	0.009235	0.000177	0.009216
	Sch.3	0.002200	0.019962	0.002941	0.021684	0.003040	0.021752	0.002843	0.021618	0.001196	0.009308	0.001238	0.009325	0.001155	0.009292
	Sch.4	0.000997	0.020124	0.001795	0.021304	0.001891	0.021368	0.001700	0.021242	0.000753	0.009334	0.000793	0.009348	0.000713	0.009320
120	Sch.1	0.000208	0.012921	0.000438	0.013303	0.000478	0.013320	0.000397	0.013288	0.000174	0.008515	0.000200	0.008522	0.000149	0.008508
	Sch.2	0.000572	0.013768	0.001020	0.014245	0.001064	0.014265	0.000976	0.014225	0.000651	0.008888	0.000678	0.008897	0.000625	0.008879
	Sch.3	0.000185	0.014082	0.000783	0.014528	0.000827	0.014548	0.000738	0.014509	0.000284	0.008919	0.000311	0.008927	0.000257	0.008911
	Sch.4	0.000505	0.013763	0.000871	0.013966	0.000916	0.013987	0.000825	0.013945	0.000415	0.008756	0.000442	0.008765	0.000388	0.008748

Graphical analysis reveals a compression between BI estimators of RT entropy under NIP and IP functions are discussed. The convergence graphs of MCMC estimates for the unknown parameters υ and ω utilizing the MH algorithm are presented. In both cases prior functions are illustrated through histograms of estimates and convergence plots. These graphs are visualized in Figures (3-10), for both NIP and IP functions. It can be observed that the BI estimators of RT entropy under the IP function exhibit superior performance compared to those under the NIP function.

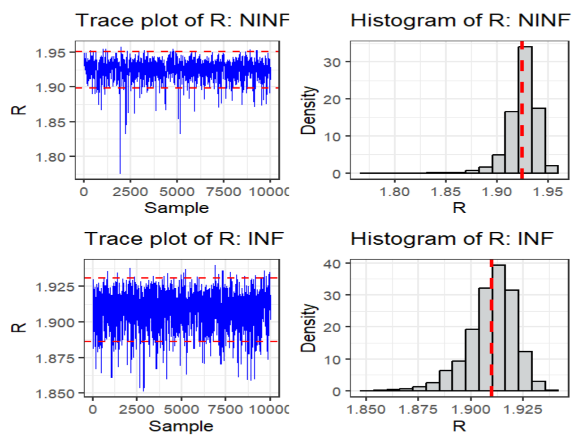

Figure 4. Convergence of MCMC for scheme 2: R given $IW(\nu = 1.5, \omega = 2.5)$ and $\alpha = 1.5, t = 0.5$

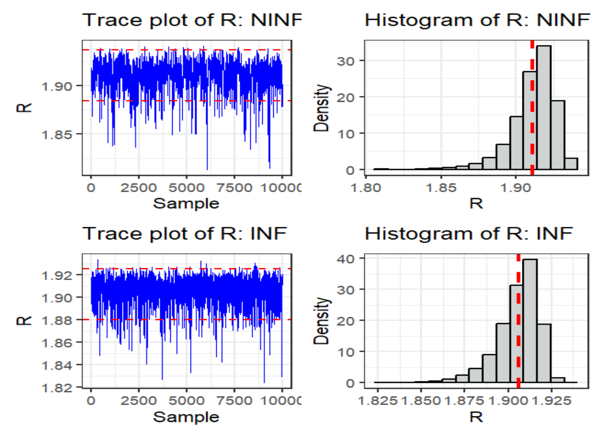

Figure 5. Convergence of MCMC for scheme 2: $\boldsymbol{R}$ given $\boldsymbol{IW}(\boldsymbol{\nu} = \mathbf{1.5}, \boldsymbol{\omega} = \mathbf{2.5})$ and $\boldsymbol{\alpha} = \mathbf{1.5}, \boldsymbol{t} = \mathbf{0.5}$

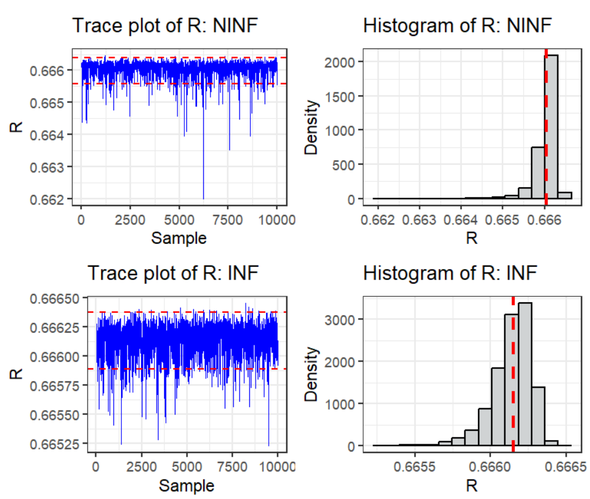

Figure 6. Convergence of MCMC for scheme 2: $\boldsymbol{R}$ given $\boldsymbol{IW}(\boldsymbol{\nu} = \mathbf{1.5}, \boldsymbol{\omega} = \mathbf{2.5})$ and $\boldsymbol{\alpha} = \mathbf{2.5}, \boldsymbol{t} = \mathbf{1.5}$

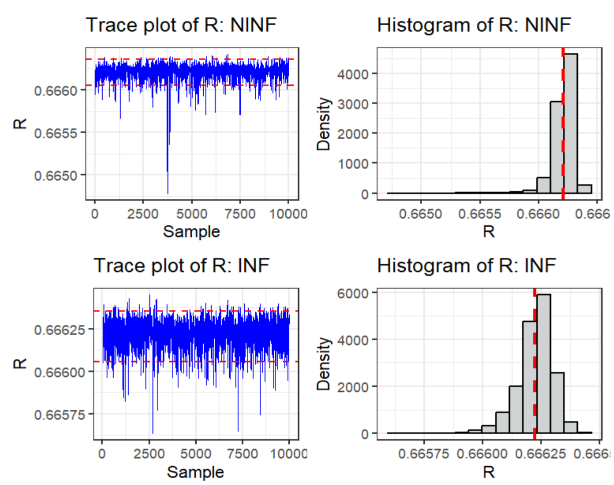

Figure 7. Convergence of MCMC for scheme 2: $\boldsymbol{R}$ given $\boldsymbol{IW}(\boldsymbol{\nu} = \mathbf{1.5}, \boldsymbol{\omega} = \mathbf{2.5})$ and $\boldsymbol{\alpha} = \mathbf{2.5}, \boldsymbol{t} = \mathbf{1.5}$

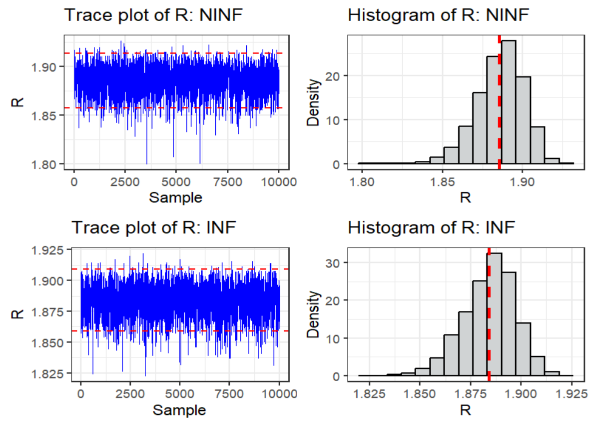

Figure 8. Convergence of MCMC for scheme 2: R given $IW(\nu = 2.5, \omega = 1.5)$ and $\alpha = 1.5, t = 0.5$

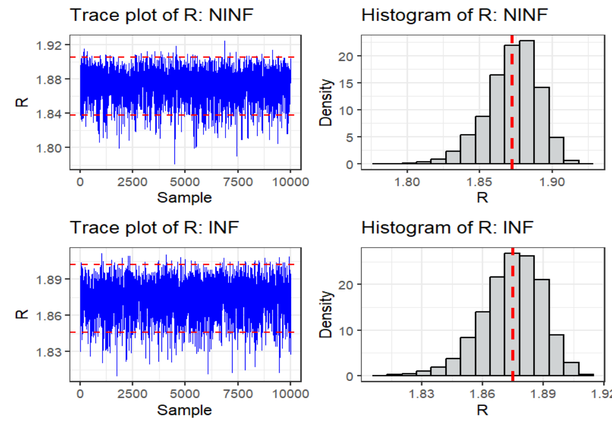

Figure 9. Convergence of MCMC for scheme 2: R given $IW(\nu = 2.5, \omega = 1.5)$ and $\alpha = 1.5, t = 0.5$

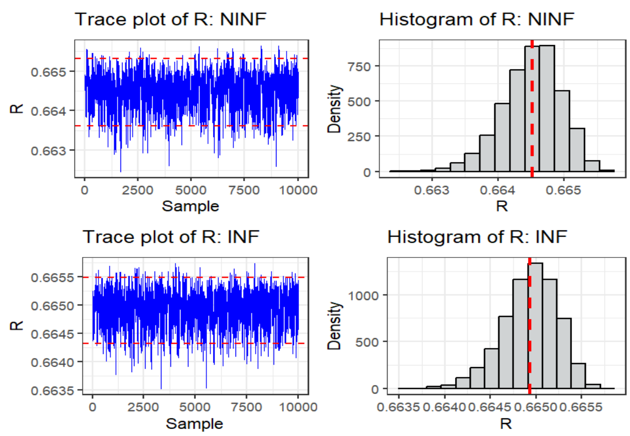

Figure 10. Convergence of MCMC for scheme 2: R given $IW(\nu = 2.5, \omega = 1.5)$ and $\alpha = 1.5, t = 0.5$

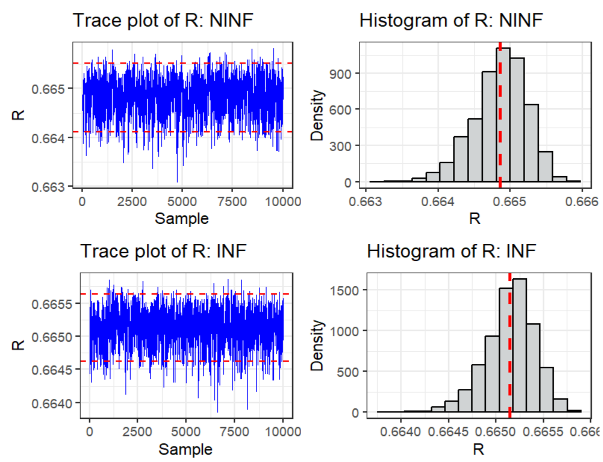

Figure 11. Convergence of MCMC for scheme 2: R given $IW(\nu = 2.5, \omega = 1.5)$ and $\alpha = 2.5, t = 0.5$

6. Conclusions

The Bayesian and non-Bayesian estimation methods are employed to estimate the residual Tsallis entropy measure for IW distribution using PCT_I sample. Two loss functions, namely, SE and LINEX are utilized in the estimation of Bayesian RT entropy in the case of NIP and IP functions.

The Bayesian estimates are evaluated using the MH algorithm based on the MCMC approach. Furthermore, BIAS and RMSE for the unknown parameters are computed in both methods of estimation. It is observed that the corresponding ML estimators do not perform as well as the BI estimators with appropriate priors. However, utilizing the NIP loss function, ML estimators exhibit competitive performance compared to BI estimators. In general, BI estimators of RT entropy measure under LINEX-II loss function demonstrate superior performance compared to other competing estimates, according to a simulation study.

Declaration of interests

The authors declare that they have no conflict of interest.

References

Abo-Eleneen, Z. A. (2011). The entropy of progressively censored samples. *Entropy*, 13(2): 437–449. doi.org/10.3390/e13020437

Al-Babtain, A. A., Elbatal I., Chesneau, C., and Elgarhy M. (2021). Estimation of different types of entropies for the Kumaraswamy distribution. *PLoS ONE*. 16(3): 1-21. doi.org/10.1371/journal.pone.0249027

Algarni, A. Elgarhy, M., Almarashi, A. M., Fayomi, A., and El-Saeed, A.R (2021). Classical and Bayesian estimation of the inverse Weibull distribution: using the progressive type-I censoring scheme. *Advances in Civil Engineering*, 5701529. 1-15. doi.org/10.1155/2021/5701529

Alkarni, S., Afify, A. Z., Elbatal, I., and Elgarhy, M. (2020). The extended inverse Weibull distribution: properties and applications. *Complexity*, 1–11. doi.org/10.1155/2020/3297693

Almohaimeed, B. (2017). On the entropy of progressive hybrid censoring schemes. *Applied Mathematics and Information Sciences*. 11(6): 1811-1814. doi.10.18576/amis/110629

Asadi, M., Ebrahimi, N., and Soofi, E. S. (2005). Dynamic generalized information measures. *Statistics and Probability Letters*, 71(1): 85–98. doi.10.1016/j.spl.2004.10.033

Asadi, M., Ebrahimi, N., Hamedani, G. G., and Soofi, E. S. (2006). Information measures for pareto distributions and order statistics. *Advances in Distribution Theory, Order Statistics, and Inference*, 207–223.

Bhaumik, D. K., Kapur, K., and Gibbons, R. D. (2009). Testing parameters of a gamma distribution for small samples. *Technometrics*, 51(3): 326–334.

Balakrishnan, N., and Aggarwala, R. (2000). Progressive Censoring Theory, Methods and Applications. Birkhäuser, Boston, Springer.

Balakrishnan, N. and Cramer, E. (2010). The art of progressive censoring: applications to reliability and quality. Springer.

Balakrishnan, N., and Cramer, E. (2014). The Art of Progressive Censoring: Applications to Reliability and Quality (Statistics for Industry and Technology), Birkhäuser, Springer New York Heidelberg Dordrecht London, USA.

Baratpour, S., Ahmadi, J., and Arghami, N. R. (2008). Characterizations based on Rényi entropy of order statistics and record values. *Journal of Statistical Planning and Inference*, 138(8): 2544–2551. doi.org/10.1016/j.jspi.2007.10.024

Basheer, A. M. (2019). Alpha power inverse Weibull distribution with reliability application. *Journal of Taibah University for Science.* 13(1): 423-432. doi.org/10.1080/16583655.2019.1588488

Basheer, A. M., Almetwally, E. M., and Okasha, H. (2020). Marshall-Olkin alpha power inverse weibull distribution: non Bayesian and Bayesian estimations. *Journal of Statistics Applications & Probability.* 10(2): 327-345. doi.10.18576/jsap/100205

Cho, Y., Sun, H., and Lee, K. (2015). Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring. *Entropy*, 17(1): 102–122. doi.org/10.3390/e17010102

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technimetrics*, 5(3): 327-339. doi.org/10.2307/1266337

Dey, S., Singh, S., Tripathi, Y.M., and Asgharzadeh, A. (2016). Estimation and prediction for a progressively censored generalized inverted exponential distribution. *Statistical Methodology*, 132: 185-202. doi.10.1016/j.stamet.2016.05.007

Ebrahimi, N. (1996). How to measure uncertainty in the residual lifetime distribution. *Sankhyā: The Indian Journal of Statistics, Series A*, 58(1): 48–56.

Elbiely, M. M., and Yousof, H. M. (2019). A new inverse Weibull distribution: properties and applications. *Journal of Mathematics and Statistics*, 15(1): 30–43. doi.10.3844/jmssp.2019.30.43

Fashandi, M., and Ahmadi, J. (2012). Characterizations of symmetric distributions based on Rényi entropy. *Statistical Probability Letter* 82: 798–804. doi.org/10.1016/j.spl.2012.01.004

Hassan, A. S., and Zaky, A. N. (2019). Estimation of entropy for inverse Weibull distribution under multiple censored data. *Journal of Taibah University for Science*, 13(1): 331–337. doi.10.1080/16583655.2019.1576493

Ibrahimi, B., Sankaran, P.G., (2005). Rényi's entropy for residual lifetime distributions. *Statistical Papers*, 46: 17–30. doi.10.1007/S00362-005-0270

Jana, N., and Bera, S. (2020). Estimation of parameters of inverse weibull distribution and application to multi-component Stress-Strength model. *Journal of Statistics.* 49(1): 1-26. doi. 10.1080/02664763.2020.1803815

Khan, M. S., Pasha, G. R., and Pasha, A. H. (2008). Theoretical analysis of inverse weibull distribution. *Wseas Transactions on Mathematics*, 7: 30-38.

Lee, K. (2017). Estimation of entropy of the inverse Weibull distribution under generalized progressive hybrid censored data. *Korean Data and Information Science Society*, 28(3):659–68. doi.org/10.3390/e17010102

Lévay, P., Nagy, S., and Pipek, J. (2005). Elementary formula for entanglement entropies of fermionic systems. *Physical Review A*, 72:1–8. doi.10.1103/PhysRevA.72.022302

Li, X., and Zhang, S. (2011). Some new results on Rényi entropy of residual life and inactivity time. *Probability in the Engineering and Informational Sciences*, 25(02): 237–250. doi.org/10.1017/S0269964810000379

Musleh, R. M., and Helu, A. (2014). Estimation of the inverse Weibull distribution based on progressively censored data: *comparative study. Reliability Engineering and System Safety*, 131: 216–227. doi.org/10.1016/j.ress.2014.07.006

Nanda, A., and Paul, P. (2006). Some results on generalized residual entropy. *Information Sciences*, 176(1): 27–47. doi.org/10.1016/j.ins.2004.10.008

Ren, H. and Hu, X. (2023). Statistical inference of the stress-strength reliability for inverse Weibull distribution under an adaptive progressive type-II censored sample. *AIMS Mathematics*, 8(12): 28465-28487. doi. 10.3934/math.20231457

Renner R., Gisin N., and Kraus, B. (2005). Information-theoretic security proof for quantum-key-distribution protocols. *Physical Review A*, 72:1–18. doi.org/10.48550/arXiv.quantph/0502064

Sadiah, A., and Abdallah, M. (2023). An extensive mathematical approach for wind speed evaluation using inverse weibull distribution. *Alexandria Engineering Journal.* 76: 775-786. doi.org/10.1016/j.aej.2023.06.076

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, 27: 379–423.

Singh, S. K., Singh, U., and Kumar, D. (2013). Bayesian estimation of parameters of inverse Weibull distribution. *Journal of Applied Statistics*, 40(7): 1597–1607. doi.10.1080/02664763.2013.789492

Song, K. S. (2001). Rényi information, log-likelihood and an intrinsic distribution measure. *Journal of Statistical Planning and Inference*, 93(1-2): 51–69. doi.org/10.1016/S0378-3758(00)00169-5

Sunoj, S. M., and Sankaran, P.G. (2012). Quantile based entropy function. *Statistics and Probability Letters*. 82(6): 1049–1053. doi.org/10.1016/j.spl.2012.02.005

Tsallis C. (1998). The role of constraints within generalized non-extensive statistics. *Physica A: Statistical Mechanics and its Applications*, 261: 547–61. doi.org/10.1016/S0378-4371(98)00437-3

Vishwakarma, P. K., Kaushik, A., Pandey, A., Singh, U., and Singh, S. K. (2018). Bayesian estimation for inverse Weibull distribution under progressive type-ii censored data with beta-binomial removals. *Austrian Journal of Statistics*, 47(1): 77–94. doi.10.17713/ajs.v47i1.578

Xu, R., and Gui, W. (2019). Entropy Estimation of inverse Weibull distribution under adaptive type-ii progressive hybrid censoring schemes. *Symmetry*, 11(12): 1-23. doi.org/10.3390/sym11121463

Zarezadeh, S., and Asadi, M. (2010). Results on residual Rényi entropy of order statistics and record values. *Information Sciences*, 180(21): 4195–4206. doi.org/10.1016/j.ins.2010.06.019