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A New Three-Parameter Inverted Exponentiated Weibull Distribution: Statistical Inference and Application

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Sine family; inverted exponentiated Weibull distribution; reversed residual life; stress–strength model; Tsallis entropy; Bayesian approach; Markov Chain Monte Carlo.

Abstract

This research introduces a novel extension of the well-established inverted exponentiated Weibull distribution. By incorporating a trigonometric sine function, we develop the sine-inverted exponentiated Weibull (SIEW) distribution, a flexible model capable of capturing a wide range of data patterns. The SIEW distribution exhibits versatility in modeling data with increasing, decreasing, reversed J-shaped, and upside-down shaped hazard rates, making it suitable for various real-world applications. To comprehensively understand the SIEW distribution, we delve into its key statistical properties and compute entropy measures such as Rényi and Tsallis. For parameter estimation, we employ both classical maximum likelihood and Bayesian estimation procedures, considering symmetric and asymmetric loss functions. Recognizing the computational challenges inherent in Bayesian estimation, we implement Markov Chain Monte Carlo techniques with independent gamma priors. The performance of the SIEW distribution is rigorously assessed through extensive simulation studies and real-world data analysis. By comparing the SIEW model to existing alternatives, we demonstrate its superior flexibility and effectiveness in modeling complex data structures. This study contributes to statistical literature by providing a new and adaptable tool for data analysis across various domains.

1. Introduction

Researchers in recent years have developed new types of flexible continuous distributions that deviate from traditional models. These distributions have been applied in a variety of fields, such as physics, biology, medicine, business, and engineering. By modifying parameters within the original cumulative distribution function (CDF), they offer greater flexibility, allowing for more accurate modeling of diverse phenomena. The introduction of new parameters has a beneficial impact on characteristics such as skewness, kurtosis, tail behavior, and central and dispersion parameters. These generalized distributions enable statisticians to address a wide range of modeling requirements and capture the complexities present in real-world data across different disciplines. The existence of numerous parameters in traditional distributions can make estimation challenging. To overcome this, researchers have explored new families of distributions, such as the

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Beta-G family by Eugene et al. (2002), the generalized family of continuous distributions by Alzaatreh et al. (2013), the Kumaraswamy-G family by Cordeiro and de Castro (2011), the alpha power transformation-G family by Mahdavi and Kundu (2017), the odd inverted Topp–Leone-G family by Hassan et al. (2022), and the Teissier-G family by Eghwerido et al. (2022), among others.

Recent studies focusing on creating trigonometric distributions have garnered significant attention. These families offer several advantages, such as striking a balance between simplicity in their definitions, which facilitates a comprehensive understanding of their mathematical characteristics, and their broad applicability in modeling various real-world datasets. These benefits are achieved through the judicious use of extensible trigonometric functions. One noteworthy development in this field is the sine- G (S-G) family of distributions. Trigonometric transformations, presented by Souza (2015), support much to enhance these distributions' versatility and flexibility. The parameters involved in these functions exhibit oscillatory behavior as their values change, and the periodic functions dictate the behavior of the distribution curve. By combining various functions with different behaviors, a more effective approach to modeling real-world phenomena is achieved, addressing the limitations of previously established generalized statistical distributions.

An example of a freshly created trigonometric distributions is the new S-G family (Kumar *et al.* (2015)), sine Topp-Leone-G family (Al-Babtain *et al.* (2020)), the exponentiated S-G family (Muhammad *et al.* (2021)), sine half-logistic inverse Rayleigh distribution (Shrahili *et al.* (2021b)), sine modified Lindley distribution (Tomy and Chesneau (2021)), sine power Lindley distribution (Almarashi (2022)), sine Burr XII distribution (Isa *et al.* (2022)), sine Lomax distribution (Mustapha *et al.* (2023)), sine truncated Lomax distribution (Elgarhy *et al.* (2023)), and sine power XLindley distribution (Alsadat et al. (2024)).

The CDF and the probability density function (PDF) of the S-G family of distributions are provided, respectively, by:

$$G(x;\varphi) = \sin\left(0.5\pi\, F(x;\varphi)\right), \quad x \in R \,, \tag{1}$$

and

$$g(x;\varphi) = 0.5\pi\, f(x;\varphi)\cos\left(0.5\pi\, F(x;\varphi)\right), \quad x \in R, \tag{2}$$

where the CDF and PDF of any baseline distribution with a vector of parameters φ are represented by the characters $F(x;\varphi)$ and $f(x;\varphi)$, respectively. The sine transformation of the CDF in these equations allows for a change in distribution skewness. By using this transformation, the distribution can be made more symmetric or more asymmetrical based on the specific requirements of statistical analyses. It is important to highlight that both $G(x;\varphi)$ and $F(x;\varphi)$ have the same number of parameters, which completely removes over-parametrization since there is no need to introduce extra parameters.

Inverted distributions are more flexible and capture unique characteristics compared to non-inverted distributions, making them useful tools in various applications. The inverted exponentiated Weibull (IEW) distribution provided by Lee *et al.* (2017), has several uses in diverse fields. It has been used in biological sciences to model species distributions, in reliability engineering to analyze complex hazard rate patterns, and in risk management and insurance to model extreme events and tail risks. The CDF of the IEW distribution is defined as:

$$F(x)=1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu} \quad , \quad x,\iota,\nu,\omega>0, \tag{3}$$

where ι,ν are shape parameters and ω is a scale parameter. The PDF related to Equation (3) is as follows:

$$f(x)=\iota\nu\omega x^{-\iota-1}\left(1-e^{-\omega x^{-\iota}}\right)^{\nu-1} e^{-\omega x^{-\iota}} \quad , \quad x,\iota,\nu,\omega>0. \tag{4}$$

The IEW distribution includes generalized inverted exponential, generalized inverted Rayleigh, inverted exponential, inverted Weibull, and inverted Rayleigh distributions as special cases.

In this research, significant progress in the formation of the S-G family by combining it with the recently developed IEW distribution is made, resulting in the creation of the sine IEW (SIEW) distribution. The major goal of the SIEW distribution is to improve the efficiency of the IEW distribution across a wide range of datasets. Further insights and motivations for this research will be unveiled as we delve deeper into the study. In particular, the SIEW distribution's PDF has special characteristics, such as a right-skewed tail and a unimodal form with both increasing and decreasing shapes. Furthermore, the hazard rate function (hrf) associated with it has an inverted pattern with alternating increases and decreases. Because of these characteristics, the SIEW distribution is a good choice for a variety of applications in a variety of fields. Many key metrics of the SIEW distribution are determined, such as the quantile function (QF), moments, moments of residual life (MRL), moments of reversed residual life (MRRL), stress-strength (SS) reliability, and information measures. We look at using maximum likelihood (ML) and Bayesian techniques for estimating the SIEW distribution's parameters. An extensive simulation study is conducted to evaluate the accuracy of the parameter estimates using various metrics. In addition, a real data set is analyzed.

The remaining sections are structured as follows: Section 2 introduces the SIEW distribution. Section 3 investigates the statistical attributes of the SIEW distribution. Section 4 focuses on parameter inference using ML and Bayesian techniques. Section 5 examines the application of SIEW distribution. Finally, some concluding remarks are presented in Section 6.

2. Methodology

We present the SIEW distribution in this section. Assume X is a continuous random variable with CDF (3) and PDF (4). Substituting Equation (3) in Equation (1) yields the new model PDF and CDF for the SIEW distribution, for $\iota,\nu,\omega>0,$ respectively, as

$$G(x;\varphi)=\sin\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right);\quad x>0, \tag{5}$$

and

$$g(x;\varphi)=0.5\pi\iota\nu\omega x^{-\iota-1}\left\{1-e^{-\omega x^{-\iota}}\right\}^{\nu-1}e^{-\omega x^{-\iota}} \times\cos\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right) ;x>0, \tag{6}$$

where $\varphi\equiv(\iota,\nu,\omega)$ is the set of parameters in which ι,ν are shape parameters and ω is a scale parameter. A random variable with CDF (5) is indicated by SIEW(ι,ν,ω), the reliability function $S(x;\varphi)$ is:

$$S(x;\varphi)=1-\sin\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right). \tag{7}$$

The hrf $h(x;\varphi)$ is:

$$h(x;\varphi)=0.5\pi e^{-\omega x^{-\iota}}\iota\nu\omega x^{-\iota-1}\left\{1-e^{-\omega x^{-\iota}}\right\}^{\nu-1}\frac{\cos\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)}{1-\sin\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)}. \tag{8}$$

The PDF (6) includes some new and existing sub models:

- For $\iota=1,$ the SIEW distribution transforms into sine generalized inverted exponential distribution (new).
- For $\iota=2,$ the SIEW distribution transforms into sine exponentiated inverted Rayleigh distribution (new).
- For $\iota=\nu=1,$ the SIEW distribution provides the sine inverted exponential distribution (see Shrahili *et al.* (2021a)).
- For $\nu=1,$ the SIEW distribution provides the sine inverted Weibull distribution (see Souza *et al.* (2019)).

- For $\nu = 1$ and $\iota = 2$, the SIEW distribution provides the sine inverted Rayleigh distribution (see Ahmadini (2022)).

The SIEW PDF and hrf graphs of various parameter combinations are displayed in Figure 1, which can show a right-skewed, unimodal, increasing, and decreasing PDF of the SIEW distribution. Additionally, hrf of the SIEW distribution could be increasing, decreasing, or upside-down.

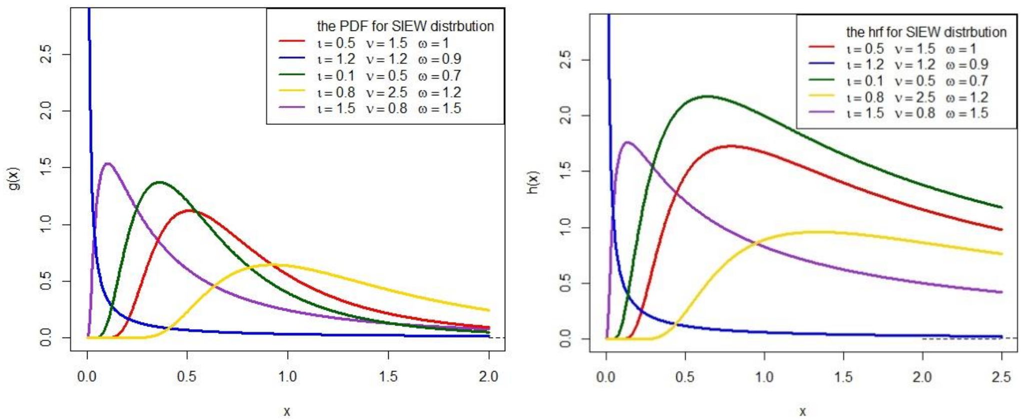

Figure 1. SIEW distribution charts for the PDF and hrf

3. Statistical Attributes

The primary attributes of the SIEW distribution, including QF, a few moment metrics, MRL, MRRL, SS reliability, and entropy measures, are considered in this section.

3.1 Quantile Function

Simply inverting the CDF in Equation (5) yields the QF of the SIEW distribution, which is specified by

$$Q(u;\varphi) = \left(-\omega^{-1} \left[\ln \left[1 - \left(1 - 2\pi^{-1} \sin^{-1}(u) \right)^{1/\nu} \right] \right] \right)^{-1/\iota}, \; 0 < u < 1, \quad \iota, \nu, \omega > 0. \qquad (9)$$

Three quartiles: first, median, and third, denoted by Q_1, Q_2 and Q_3 can be obtained from Equation (9) when u = 0.25, 0.5 and 0.75.

3.2 Stress-Strength Reliability

In this part, the SS reliability parameter for the SIEW distribution is determined. The SS reliability is defined by $R = P(Y < X)$. In a basic situation, a system with a random strength value (X), is exposed to a random stress level (Y). If the stress level exceeds the system's ability (strength), it will fail. Suppose that X is the strength random variable having PDF $g(\iota, \nu_1, \omega)$ and Y is the stress random variable having CDF $G(\iota, \nu_2, \omega)$, where Y and X are assumed to be independent. Then, the SS reliability parameter is obtained as follows:

$$R=\frac{\pi}{2}\iota\nu_1\omega\int_0^\infty x^{-\iota-1}\left\{1-e^{-\omega x^{-\iota}}\right\}^{\nu_1-1}\cos\left(\frac{\pi}{2}\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu_1}\right]\right)$$
$$e^{-\omega x^{-\iota}}\sin\left(\frac{\pi}{2}\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu_2}\right]\right)dx.$$

Let $u=1-e^{-\omega x^{-\iota}} \Rightarrow du=-\omega\iota x^{-\iota-1}e^{-\omega x^{-\iota}}dx,$ then

$$R=\nu_1\frac{\pi}{2}\int_0^1 u^{\nu_1-1}\cos\left(\frac{\pi}{2}\left[1-u^{\nu_1}\right]\right)\sin\left(\frac{\pi}{2}\left[1-u^{\nu_2}\right]\right)du. \tag{10}$$

The challenge arises from being unable to directly calculate this integral. Therefore, a different technique based on a series expansion will be explored. To do this, it is important to remember the sine and cosine series expansions, for all real values of z, which are expressed as follows:

$$\sin(z)=\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)!}z^{2p+1}, \tag{11}$$

and,

$$\cos(z)=\sum_{i=0}^{\infty}\frac{(-1)^i}{2i!}z^{2i}. \tag{12}$$

Using the expansions given in Equations (11) and (12) in Equation (10) give

$$\begin{aligned}R&=\sum_{i,p=0}^{\infty}\theta_{i,p}(\nu_1,\nu_2)\nu_1\int_0^1 u^{\nu_1-1}(1-u^{\nu_1})^{2i}(1-u^{\nu_2})^{2p+1}du\\&=\sum_{i,p=0}^{\infty}\sum_{d=0}^{2p+1}(-1)^d\theta_{i,p}(\nu_1,\nu_2)\binom{2p+1}{d}B\left(2i+1,\frac{\nu_2 d}{\nu_1}+1\right),\end{aligned} \tag{13}$$

where $\theta_{i,p}(\nu_1,\nu_2)=\frac{(-1)^{i+p}}{(2i)!(2p+1)!}\left(\frac{\pi}{2}\right)^{2(i+p+1)}$, and $B(.,.)$ is the beta function.

3.3 Information Measures

This part focuses on entropy measures, including Rényi entropy (RE) and Tsallis entropy (TE), which give an idea of the entire quantity of data in the system. The RE, proposed by Rényi (1961), plays an important role in this context. The RE of X is defined as

$$\Phi(\rho)=(1-\rho)^{-1}\log\left[\int_0^\infty\left(g(x;\varphi)\right)^{\rho}dx\right];\quad x,\varphi,\rho>0,\ \rho\neq1, \tag{14}$$

where ρ is the entropy order. To obtain $\Phi(\rho)$ of the SIEW distribution, we must obtain $\left(g(x;\varphi)\right)^{\rho}$, as follows:

$$\left(g(x;\varphi)\right)^{\rho}=\left(0.5\pi\iota\nu\omega\right)^{\rho}x^{-\rho\iota-\rho}e^{-\omega\rho x^{-\iota}}\left(1-e^{-\omega x^{-\iota}}\right)^{\rho\nu-\rho}\times\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right). \quad (15)$$

Using the binomial theory, then $\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)$ is expanded as follows:

$$\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)=\sum_{j=0}^{\infty}\binom{\rho}{j}\left\{\cos\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)-1\right\}^{j},$$

which can be written as:

$$\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)=1+\sum_{j=1}^{\infty}\binom{\rho}{j}\left\{\cos\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)-1\right\}^{j}. \quad (16)$$

Using Expansion (12) in Equation (16) provides

$$\begin{aligned}&\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)\\&=1+\sum_{j=1}^{\infty}\binom{\rho}{j}\left[\sum_{i=0}^{\infty}\frac{(-1)^{i}(0.5\pi)^{2i}}{(2i)!}\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2i}-1\right]^{j}\\&=1+\sum_{j=1}^{\infty}\binom{\rho}{j}\left[\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(0.5\pi)^{2(k+1)}}{(2(k+1))!}\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2(k+1)}\right]^{j},\end{aligned}$$

where $i=k+1, k\geq 0,$ k is a positive integer, $i\geq 1,$ then $\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)$ is expanded as follows:

$$\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)$$
$$=1+\sum_{j=1}^{\infty}\binom{\rho}{j}\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2j}\left[\sum_{k=0}^{\infty}z_k\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2k}\right]^{j}, \qquad (17)$$

where $z_k=\dfrac{(\pi/2)^{2(k+1)}(-1)^{k+1}}{(2(k+1))!}$.

Since $j \geq 1$, for a given power series of the last term in Equation (17).

$$\left(\sum_{k=0}^{\infty}z_k x^k\right)^{j}=\sum_{k=0}^{\infty}c_k x^k,$$

where j is a positive integer, $c_0=z_0^j, c_j=(\frac{1}{m}z_0)\sum_{k=1}^{m}(k\,j-m+k)\,z_k\;c_{m-k}, m\geq 1.$

Then

$$\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)=1+\sum_{j=1}^{\infty}\binom{\rho}{j}\sum_{k=0}^{\infty}z_m^{*}\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2(m+j)},$$

where $z_0=z_0^j$ and $z_m^{*}=(\frac{1}{m}z_0)\sum_{k=1}^{m}(k\,j-m+k)\;z_k\;z_{m-k}^{*}$, $m\geq 1.$

Hence,

$$\cos^{\rho}\left(0.5\pi\left[1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right]\right)=1+\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}z_{k,j}^{**}\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2(m+j)}, \qquad (18)$$

where $z_{k,j}^{**}=\binom{\rho}{j}z_k^{*}$. For more details see (Gradshteyn *et al.* (2007)). Substituting Equation (18) in Equation (15) give:

$$\left(g(x;\varphi)\right)^{\rho}=\left(0.5\pi\iota\nu\omega\right)^{\rho}x^{-\rho\iota-1}e^{-\omega\rho x^{-\iota}}\left(1-e^{-\omega x^{-\iota}}\right)^{\rho\nu-\rho}+\left(0.5\pi\iota\nu\omega\right)^{\rho}$$
$$\times\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}z_{k,j}^{**}x^{-\rho\iota-1}e^{-\omega\rho x^{-\iota}}\left(1-e^{-\omega x^{-\iota}}\right)^{\rho\nu-\rho}\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2(k+j)}.$$

Applying the binomial expansion in the terms $\left(1-\left(1-e^{-\omega x^{-\iota}}\right)^{\nu}\right)^{2(k+j)}$ and $\left(1-e^{-\omega x^{-\iota}}\right)^{\rho\nu-\rho}$. Then $\left(g(x;\varphi)\right)^{\rho}$ is as follows:

$$\left(g(x;\varphi)\right)^{\rho}=\left(0.5\pi\iota\nu\omega\right)^{\rho}\sum_{u=0}^{\infty}\binom{\rho(\nu-1)}{u}(-1)^{u}\,x^{-\rho\iota-1}e^{-\omega\rho u\,x^{-\iota}}+\left(0.5\pi\iota\nu\omega\right)^{\rho}\times\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{2(k+j)}\binom{2(k+j)}{n}(-1)^{n}x^{-\rho\iota-1}z_{k,j}^{**}\,e^{-\omega\rho\,x^{-\iota}}\left(1-e^{-\omega x^{-\iota}}\right)^{\rho(\nu-1)+\nu n}.$$

Again, the binomial expansion is in the term $\left(1-e^{-\omega x^{-\iota}}\right)^{\rho(\nu-1)+\nu n}$, which leads to

$$\left(g(x;\varphi)\right)^{\rho}=\left(0.5\pi\iota\nu\omega\right)^{\rho}\sum_{u=0}^{\infty}\binom{\rho(\nu-1)}{u}(-1)^{u}\,x^{-\rho\iota-1}e^{-\omega\rho u\,x^{-\iota}}+\left(0.5\pi\iota\nu\omega\right)^{\rho}\times\sum_{j=1}^{\infty}\sum_{k,t=0}^{\infty}\sum_{n=0}^{2(k+j)}\binom{2(k+j)}{n}\binom{\rho(\nu-1)+\nu n}{t}(-1)^{n}x^{-\rho\iota-1}z_{k,j}^{**}\,e^{-\omega\rho t\,x^{-\iota}}. \quad (19)$$

Substituting Equation (19) into Equation (14) , then $\Phi(\rho)$ becomes as follows:

$$\Phi(\rho)=(1-\rho)^{-1}\left(\Gamma\left(\frac{\rho(\iota+1)-1}{\iota}\right)\left\{\begin{array}{l}\sum_{u=0}^{\infty}\psi_{u}(\omega,\iota,\nu)\left(\omega(\rho+u)\right)^{\frac{1-\rho(\iota+1)}{\iota}}\\+\sum_{j=1}^{\infty}L_{k,n,t}(\nu,\omega,\iota)\left(\omega(t+\rho)\right)^{\frac{1-\rho(\iota+1)}{\iota}}\end{array}\right\}\right),$$

where $\Gamma(n)$ is the gamma function (GF), $\psi_{u}(\omega,\iota,\nu)=\binom{\rho(\nu-1)}{u}(-1)^{u}\left(\frac{\pi}{2}\nu\omega\right)^{\rho}\iota^{\rho-1}$,

and,

$$L_{k,n,t}(\nu,\omega,\iota)=\sum_{k,t=0}^{\infty}\sum_{n=0}^{2(k+j)}z_{k,j}^{**}(-1)^{n+t}\iota^{\rho-1}\left(\frac{\pi}{2}\iota\nu\omega\right)^{\rho}\binom{\rho(\nu-1)+\nu n}{t}\binom{2(k+j)}{n}.$$

Furthermore, the TE of the SIEW distribution is obtained according to Tsallis (1988) as follows:

$$\Omega(\rho)=\left[1-\int_0^{\infty}\left(g(x;\varphi)\right)^{\rho}dx\right](1-\rho)^{-1},\quad x,\varphi,\rho>0,\ \rho\neq 1$$

$$=\left[1-\left(\Gamma\left(\frac{\rho(\iota+1)-1}{\iota}\right)\begin{Bmatrix}\sum_{u=0}^{\infty}\psi_u(\omega,\iota,\nu)\left(\omega(\rho+u)\right)^{\frac{1-\rho(\iota+1)}{\iota}}\\+\sum_{j=1}^{\infty}L_{k,n,t}(\nu,\omega,\iota)\left(\omega(t+\rho)\right)^{\frac{1-\rho(\iota+1)}{\iota}}\end{Bmatrix}\right)\right](1-\rho)^{-1}.$$

Table 1 displays various numerical values of $\Phi(\rho)$ and $\Omega(\rho)$ for certain parameter values, (a) $(\iota=5,\nu=2,\omega=1)$, (b) $(\iota=5,\nu=3,\omega=1)$,(c) $(\iota=5,\nu=5,\omega=1)$, (d) $(\iota=10,\nu=2,\omega=1)$, (e) $(\iota=10,\nu=3,\omega=1)$, (f) $(\iota=10,\nu=2,\omega=1)$. The findings indicate that as the value of ρ increases, the values of both entropy measures decrease. Also, the values of $\Omega(\rho)$ are larger than those of $\Phi(\rho)$ for $\rho=0.5$ and $\rho=0.9$, the inverse holds for $\rho=2$.

Table 1. Entropy measures of the SIEW distribution for $\rho=0.5, 0.9$, and 2.

ρ	**Measures**	**(a)**	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**
0.5	$\Phi(\rho)$	-0.5886	-0.8156	-1.0580	-1.2481	-1.4465	-1.6612
	$\Omega(\rho)$	-0.5099	-0.6698	-0.8216	-0.9285	-1.0296	-1.1284
0.9	$\Phi(\rho)$	-0.7946	-1.0047	-1.2348	-1.4394	-1.6273	-1.8345
	$\Omega(\rho)$	-0.7639	-0.9559	-1.1616	-1.3406	-1.5018	-1.6761
2	$\Phi(\rho)$	-0.9996	-1.1996	-1.4225	-1.6360	-1.8176	-2.0206
	$\Omega(\rho)$	-1.7171	-2.3187	-3.1473	-4.1345	-5.1569	-6.5429

3.4 Linear Expansion

The key linear combination of the PDF for the SIEW distribution will be discussed here. Hence, by substituting Equation (12) in Equation (6), gives:

$$g(x;\varphi)=\sum_{i=0}^{\infty}\left[1-\left\{1-e^{-\omega x^{-\iota}}\right\}^{\nu}\right]^{2i}\frac{(-1)^i\pi^{2i+1}\iota\nu\omega e^{-\omega x^{-\iota}}x^{-\iota-1}}{(2i)!\,2^{2i+1}}\left[1-e^{-\omega x^{-\iota}}\right]^{\nu-1}.\quad(20)$$

Applying the binomial expansion in Equation (20), then $g(x;\varphi)$ takes the form:

$$g(x;\varphi)=\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\binom{2i}{j}\frac{(-1)^{i+j}\pi^{2i+1}}{(2i)!\,2^{2i+1}}e^{-\omega x^{-\iota}}\iota\nu x^{-\iota-1}\omega\left\{1-e^{-\omega x^{-\iota}}\right\}^{\nu j+\nu-1}.\quad(21)$$

Again, the binomial expansion is used in Equation (21), which leads to

$$g(x;\varphi)=\sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{i+j+k}\pi^{2i+1}}{(2i)!\,2^{2i+1}}\binom{\nu j+\nu-1}{k}\binom{2i}{j}x^{-\iota-1}\omega\iota\nu e^{-\omega(k+1)x^{-\iota}}.$$

Thus, a simplified form of the PDF of the SIEW distribution is given by:

$$g(x;\varphi) = \sum_{i,k=0}^{\infty} \eta_{i,j,k}(\nu,\omega)\, e^{-\omega(k+1)x^{-\iota}}\, \iota\, x^{-(\iota+1)}, \tag{22}$$

where $\eta_{i,j,k}(\nu,\omega) = \sum_{j=0}^{2i} \binom{2i}{j} \nu\,\omega \frac{(-1)^{i+j+k}\pi^{2i+1}}{(2i)!\,2^{2i+1}} \binom{\nu j+\nu-1}{k}.$

3.5 Moments

The r^{th} moment for X with PDF (22) is computed as:

$$\begin{aligned} \mu'_r &= \sum_{i,k=0}^{\infty} \eta_{i,j,k}(\nu,\omega)\,\iota \int_0^{\infty} x^{r-\iota-1} e^{-\omega(k+1)x^{-\iota}}\, dx \\ &= \sum_{i,k=0}^{\infty} \eta_{i,j,k}(\nu,\omega)\,\Gamma\left(1-\frac{r}{\iota}\right)\left(\omega\,(k+1)\right)^{\frac{r}{\iota}-1}, \quad \iota > r. \end{aligned} \tag{23}$$

Thus, the moments of X have simple expressions and can be calculated easily. Table 2 provides numerical values of the first four moments, variance (σ^2), skewness (SK), and kurtosis (KU) of the SIEW distribution for the same selected parameter values.

Table 2. Some moments of the SIEW distribution.

Measures	(a)	(b)	(c)	(d)	(e)	(f)
μ'_1	0.91131	0.87114	0.83185	0.95293	0.93219	0.91128
μ'_2	0.84263	0.76659	0.69672	0.91131	0.87114	0.83185
μ'_3	0.79106	0.68156	0.58755	0.87467	0.81616	0.76064
μ'_4	0.75457	0.61238	0.49891	0.84263	0.76659	0.69672
σ^2	0.01215	0.00770	0.00475	0.00324	0.00217	0.00142
SK	0.75251	0.50985	0.28417	0.53362	0.34198	0.15350
KU	4.30784	3.64226	3.24702	3.69968	3.34010	3.12535

According to Table 2, the values of the moments, variance, skewness, and kurtosis measures of the SIEW distribution decrease as the values of ι, and ν increase. Moreover, the SIEW distribution tends to be right-skewed and leptokurtic (KU>3). Furthermore, using Equation (23), the moment-generating function can be defined as

$$M_X(t) = \sum_{r,i,k=0}^{\infty} \frac{t^r}{r!} \eta_{i,j,k}(\nu,\omega)\,\Gamma\left(1-\frac{r}{\iota}\right)\left(\omega\,(k+1)\right)^{\frac{r}{\iota}-1}; \quad \iota > r.$$

Incomplete moments play a role when estimating measures of inequality such as the Lorenz and the Bonferroni curves. The r^{th} incomplete moment of the SIEW distribution is as follows:

$$T_r(t) = \sum_{i,k=0}^{\infty} \eta_{i,j,k}(\nu,\omega)\,\iota \int_0^t x^{r-\iota+1} e^{-\omega(k+1)x^{-\iota}} dx$$
$$= \sum_{i,k=0}^{\infty} \eta_{i,j,k}(\nu,\omega)\Gamma\left(1-\frac{r}{\iota}, \omega\,(k+1)t^{-\iota}\right)\left(\omega\,(k+1)\right)^{\frac{r}{\iota}-1}; \quad \iota > r, \tag{24}$$

where $\Gamma(n,t)$ is the upper incomplete GF. The first incomplete moment for $r = 1$ in Equation (24) is particularly useful for analysis. Inequality measures, such as the Bonferroni and Lorenz curves, are commonly applied in various fields. Both the Lorenz curve $L_F(t) = T_1(t)/E(X)$ and the Bonferroni curve $B_F(t) = L_F(t)/G(t;\varphi)$ are also generated for this purpose.

3.6 Residual and Reversed Residual Life

Residual life functions, particularly the mean residual life function, failure rate function, and left-censored mean, applications are found across diverse domains, encompassing engineering, quality control, and life testing, among others. Based on the distribution's moments, the mean residual life function estimates the remaining lifespan or time until an event happens. These functions are used to compute asymptotic distributions of order statistics. They help in forecasting the remaining usable life of components, testing item reliability, and examining reliability characteristics. The study of residual life functions advances probability and statistics, allowing for more informed decision-making and increasing the reliability of designed systems. The nth MRL, represented by $M_n(t) = E\left[(X-t)^n | X > t\right]$ of X is defined by:

$$M_n(t) = \frac{1}{S(t;\varphi)} \int_t^{\infty} (x-t)^n\, g(x;\varphi)\, dx. \tag{25}$$

Using the binomial expansion in Equation (25), $M_n(t)$ is given by:

$$M_n(t) = \frac{1}{S(t;\varphi)} \sum_{i,k=0}^{\infty} E_{i,j,k}(\nu,\omega)\,\iota \int_t^{\infty} e^{-\omega(k+1)x^{-\iota}} x^{h-\iota-1} dx$$
$$= \frac{1}{S(t;\varphi)} \sum_{i,k=0}^{\infty} E_{i,j,k}(\nu,\omega)\left(\omega\,(k+1)\right)^{\frac{h}{\iota}-1} \Gamma\left(1-\frac{h}{\iota}, \left(\omega\,(k+1)\right)t^{-\iota}\right); \quad \iota > h, \tag{26}$$

where $E_{i,j,k}(\nu,\omega) = \sum_{h=0}^{n} \binom{n}{h} (-1)^{n-h} \eta_{i,j,k}(\nu,\omega) t^{n-h}$, and $\gamma(n,t)$ is the lower incomplete GF. The mean of the residual life related to the SIEW distribution is calculated by computing the expected value of the remaining lifetime following a time period of *t*, it is obtained by setting $n = 1$ in Equation (26). However, the nth MRRL of *X* is determined by:

$$W_n(t)=E\left(\left(t-X\right)^n \middle| X\le t\right)=\frac{1}{G(t;\varphi)}\int_0^t g(x;\varphi)\left(t-x\right)^n dx$$

$$=\frac{1}{G(t;\varphi)}\sum_{s=0}^{n} C_{i,j,k}(\nu,\omega)\left(\omega\,(k+1)\right)^{\frac{s}{\iota}-1}\Gamma\left(1-\frac{s}{\iota},\omega\,(k+1)t^{-\iota}\right);\quad \iota>s,$$

where $C_{i,j,k}(\nu,\omega)=\sum_{i,k=0}^{\infty}\eta_{i,j,k}(\nu,\omega)\binom{n}{s}(-1)^{n-s}\,t^{n-s}$, and $\Gamma(n,t)$ is the upper incomplete GF. Setting n = 1, the mean of the reversed residual life associated with the SIEW distribution is determined.

4. Statistical Inference

In this section, the parameter estimation for the SIEW distribution is conducted using both ML and Bayesian methods. In the Bayesian approach several different loss functions (LFs) are used, including the minimum expected LF (MLF), the linear exponential (LINEX) LF, and the squared error LF (SELF). This section provides an exact estimation of the SIEW distribution parameters as well as an analysis of the results of the various estimation strategies.

4.1 Maximum Likelihood Estimation

Consider a simple random sample $\underline{x}=(x_1, x_2, \ldots, x_n)$ of size n drawn from a population having a SIEW distribution given by Equation (6) with unknown parameter vector $\varphi=(\iota,\nu,\omega)^T$. Then the likelihood function $l^{\bullet}(\underline{x}|\varphi)$ takes the form

$$l^{\bullet}(\underline{x}|\varphi)=\left(0.5\pi\iota\nu\,\omega\right)^n e^{-\omega\sum_{i=1}^{n}x_i^{-\iota}}\prod_{i=1}^{n}x_i^{-\iota-1}\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu-1}\cos\left(0.5\pi\left(1-\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu}\right)\right).\quad (27)$$

The log likelihood function for φ, say $\log l^{\bullet}$, is given by

$$\log l^{\bullet}=n\ln\left(0.5\pi\right)+n\ln\left(\iota\right)+n\ln\left(\nu\right)+n\ln\left(\omega\right)-\left(\iota+1\right)\sum_{i=1}^{n}\ln(x_i)-\omega\sum_{i=1}^{n}x_i^{-\iota}$$
$$+(\nu-1)\sum_{i=1}^{n}\ln\left(\mathrm{A}_i(\iota,\omega)\right)+\sum_{i=1}^{n}\ln\cos\left(0.5\pi\left(1-\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu}\right)\right),$$

where $\mathrm{A}_i(\iota,\omega)=1-e^{-\omega x_i^{-\iota}}$. The ML estimators of $\iota,\nu,$ and ω are, respectively, obtained by maximizing this log-likelihood function. We can determine it by considering the derivative of $\log l^{\bullet}$ as follows:

$$\frac{\partial \log l^{\bullet}}{\partial \iota}=\frac{n}{\iota}-\sum_{i=1}^{n}\ln(x_i)+\omega\sum_{i=1}^{n}x_i^{-\iota}\ln(x_i)-\sum_{i=1}^{n}\frac{(\nu-1)\,\omega x_i^{-\iota}\ln(x_i)}{e^{\omega x_i^{-\iota}}-1}$$
$$-\sum_{i=1}^{n}0.5\pi\nu\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu-1}e^{-\omega x_i^{-\iota}}\omega x_i^{-\iota}\ln(x_i)\tan\left(0.5\pi\left(1-\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu}\right)\right),$$

$$\frac{\partial \log l^{\bullet}}{\partial \nu}=\frac{n}{\nu}+\sum_{i=1}^{n}\ln\left(\mathrm{A}_i(\iota,\omega)\right)+\sum_{i=1}^{n}0.5\pi\tan\left[0.5\pi\left(1-\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu}\right)\right]\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu}\ln\left(\mathrm{A}_i(\iota,\omega)\right),$$

and,

$$\frac{\partial \log l^{\bullet}}{\partial \omega}=\frac{n}{\omega}-\sum_{i=1}^{n}x_i^{-\iota}+\sum_{i=1}^{n}\frac{(\nu-1)x_i^{-\iota}e^{-\omega x_i^{-\iota}}}{\mathrm{A}_i(\iota,\omega)}+\sum_{i=1}^{n}0.5\pi\,\nu x_i^{-\iota}\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu-1}e^{-\omega x_i^{-\iota}}$$
$$\times\tan\left(0.5\pi\left(1-\left(\mathrm{A}_i(\iota,\omega)\right)^{\nu}\right)\right).$$

Then the ML estimators $\hat{\iota},\hat{\nu}$,and $\hat{\omega}$ of $\iota,\nu,$ and ω are obtained by solving $\frac{\partial \log l^{\bullet}}{\partial \iota}=0,$ $\frac{\partial \log l^{\bullet}}{\partial \nu}=0,$ and $\frac{\partial \log l^{\bullet}}{\partial \omega}=0,$ numerically by iterative techniques with R software.

4.2 Bayesian Estimation

A popular statistical approach to inference that deviates from conventional methods is the Bayesian methodology. It involves giving uncertainty to distribution parameters using joint prior distribution and different symmetric and asymmetric LFs.

4.2.1 Prior Information

To discuss the Bayesian estimation approach, in our assumption, we consider the parameters $\iota,\nu,$ and ω to be independently distributed according to the gamma distribution. Let ai, and bi denote the shape and scale hyper-parameters of the gamma priors for $\iota,\nu,$ and $\omega,$ respectively, where i =1, 2, 3,

$$\pi(\varphi)\propto e^{-(\iota b_1+\nu b_2+\omega b_3)}\,\iota^{a_1-1}\nu^{a_2-1}\omega^{a_3-1},\iota,\nu,\omega>0;a_i,b_i>0;i=1,2,3. \qquad (28)$$

The hyperparameters will be determined using informative priors. The gamma distribution is used as the prior with shape and scale parameters serving as hyper-parameters. The hyper-parameters are selected by utilizing information from the MLEs of the parameters $\iota,\nu,$ and ω in the SIEW distribution, along with details from the gamma distribution. This method is referred to as the elicitation of hyper-parameters proposed by Dey et al. (2014)). By equating $\hat{\iota},\hat{\nu},$ and $\hat{\omega}$ with the mean and variance of the gamma prior distributions, the corresponding means and variances are obtained:

$$\frac{a_i}{b_i} = N^{-1}\sum_{i=1}^{N}\hat{\varphi}^i,\ \frac{a_i}{b_i^2} = (N-1)^{-1}\sum_{i=1}^{N}\left(\hat{\varphi}^i - N^{-1}\sum_{i=1}^{N}\hat{\varphi}^i\right)^2,$$

where $\hat{\varphi} \equiv (\hat{\iota}, \hat{\nu}, \hat{\omega})$ and N is a total iteration of simulation. Now, on solving the above two equations, the estimated hyper-parameters can be written as:

$$a_i = \frac{\left(N^{-1}\sum_{i=1}^{N}\hat{\varphi}^i\right)^2}{(N-1)^{-1}\sum_{i=1}^{N}\left(\hat{\varphi}^i - N^{-1}\sum_{i=1}^{N}\hat{\varphi}^i\right)^2},\ \text{and}\ b_i = \frac{N^{-1}\sum_{i=1}^{N}\hat{\varphi}^i}{(N-1)^{-1}\sum_{i=1}^{N}\left(\hat{\varphi}^i - N^{-1}\sum_{i=1}^{N}\hat{\varphi}^i\right)^2}. \tag{29}$$

Equation (29) is solved to calculate the estimated hyper-parameters, the technique discussed in the following subsections can be applied.

4.2.2 Posterior Distribution

Bayesian estimators are proposed for the same unknown parameters. Both asymmetric LFs (LINEX and MLF) and the symmetric LF (SELF) are employed. In addition, the independent gamma priors in Equation (28) are used. The joint posterior distribution is obtained by combining the likelihood function (27) with the joint prior function (29). The joint posterior density function is formed as:

$$\pi(\varphi|\underline{x}) = J^{-1}\iota^{n+a_1-1}\,e^{-(\iota b_1+\nu b_2+\omega b_3)}\nu^{n+a_2-1}e^{-(\iota+1)\sum_{i=1}^{n}\ln x_i}\,e^{-\omega\sum_{i=1}^{n}x_i^{-\iota}}$$
$$\times\omega^{n+a_3-1}e^{(\nu-1)\sum_{i=1}^{n}\ln \mathrm{A}_i(\iota,\omega)}\,e^{\sum_{i=1}^{n}\ln\cos\left(\left(\frac{\pi}{2}\left[1-(\mathrm{A}_i(\iota,\omega))^{\nu}\right]\right)\right)}, \tag{30}$$

where

$$J = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\iota^{n+a_1-1}\nu^{n+a_2-1}\omega^{n+a_3-1}\,e^{-(\iota+1)\sum_{i=1}^{n}\ln x_i}\,e^{-\omega\sum_{i=1}^{n}x_i^{-\iota}}\,e^{(\nu-1)\sum_{i=1}^{n}\ln(\mathrm{A}_i(\iota,\omega))}\,e^{-(\iota b_1+\nu b_2+\omega b_3)}$$
$$\times e^{\sum_{i=1}^{n}\ln\cos\left(\frac{\pi}{2}\left[1-(\mathrm{A}_i(\iota,\omega))^{\nu}\right]\right)}d\iota\, d\nu\, d\omega.$$

Under the SELF, the Bayesian estimator of φ is as follows:

$$\hat{\varphi} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\varphi\,\pi(\varphi|\underline{x})\,d\iota\, d\nu\, d\omega. \tag{31}$$

Under the LINEX, the Bayesian estimator of φ is as follows:

$$\hat{\hat{\varphi}} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-c\varphi}\, \pi(\varphi|\underline{x})\, d\iota\, d\nu\, d\omega. \tag{32}$$

The Bayesian estimator of φ under MLF (Tummala and Sathe (1978)) is as follows:

$$\widehat{\varphi} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\varphi^{-1}}{\varphi^{-2}}\, \pi(\varphi|\underline{x})\, d\iota\, d\nu\, d\omega. \tag{33}$$

The Bayes estimators for parameters $\iota, \nu,$ and ω cannot be obtained explicitly because of the complexity of the LFs involved, as seen in Equation (31) to Equation (33). To alleviate this constraint, one way is to use the Bayes Markov Chain Monte Carlo (MCMC) algorithm. Taking samples from the conditional posterior distribution is included. The posterior distribution produced by the samples may be used to estimate the Bayes estimators for the parameters $\iota, \nu,$ and ω using MCMC.

4.2.3 Markov Chain Monte Carlo Algorithm

Due to the mathematical difficulties of computing the integrals necessary in Equations (31) to (33), the MCMC technique will be applied to solve this problem. The Gibbs sampling and Metropolis-Hastings (MH) samplers are two major MCMC algorithms. In this context, similar to acceptance-rejection sampling, the MH technique considers a candidate value produced by proposal distribution as a normal distribution for every iteration of the method. From Equation (30), the conditional densities of $\iota, \nu,$ and ω are given by:

$$\pi_1(\iota|\nu, \omega, \underline{x}) \propto \iota^{n+a_1-1} e^{-\iota\sum_{i=1}^{n}\ln x_i - \omega\sum_{i=1}^{n} x_i^{-\iota} - \iota b_1}\, e^{(\nu-1)\sum_{i=1}^{n}\ln(\mathrm{A}_i(\iota,\omega))}\, e^{\sum_{i=1}^{n}\ln\cos\left(\frac{\pi}{2}\left[1-(\mathrm{A}_i(\iota,\omega))^{\nu}\right]\right)},$$

$$\pi_2(\nu|\iota, \omega, \underline{x}) \propto \nu^{n+a_2-1} e^{-\nu\left(b_2 - \sum_{i=1}^{n}\ln(\mathrm{A}_i(\iota,\omega))\right)}\, e^{\sum_{i=1}^{n}\ln\cos\left(\frac{\pi}{2}\left[1-(\mathrm{A}_i(\iota,\omega))^{\nu}\right]\right)},$$

and

$$\pi_3(\omega|\iota, \nu, \underline{x}) \propto \omega^{n+a_3-1} e^{-\omega(b_3 + \sum_{i=1}^{n} x_i^{-\iota})}\, e^{(\nu-1)\sum_{i=1}^{n}\ln(\mathrm{A}_i(\iota,\omega))}\, e^{\sum_{i=1}^{n}\ln\cos\left(\frac{\pi}{2}\left[1-(\mathrm{A}_i(\iota,\omega))^{\nu}\right]\right)}.$$

The steps of the MH algorithm are described below:

1. Start by initial guess $\left(\iota^{(0)}, \nu^{(0)}, \omega^{(0)}\right) = \left(\hat{\iota}, \hat{\nu}, \hat{\omega}\right)$.
2. Set $j = 1$.
3. Apply the MH algorithm to generate ι^*, ν^*, and ω^* from symmetric proposal distributions which are normal for ι, ν ,and ω.
4. Obtain

$$\Theta_{\iota}=\min\left\{1,\frac{\pi\left(\iota^{*}\middle|\nu^{(j-1)},\omega^{(j-1)},\underline{x}\right)}{\pi\left(\iota^{(j-1)}\middle|\nu^{(j-1)},\omega^{(j-1)},\underline{x}\right)}\right\},\ \Theta_{\nu}=\min\left\{1,\frac{\pi\left(\nu^{*}\middle|\iota^{(j-1)},\omega^{(j-1)},\underline{x}\right)}{\pi\left(\nu^{(j-1)}\middle|\iota^{(j-1)},\omega^{(j-1)},\underline{x}\right)}\right\},$$

$$\Theta_{\omega}=\min\left\{1,\frac{\pi\left(\omega^{*}\middle|\nu^{(j-1)},\omega^{(j-1)},\underline{x}\right)}{\pi\left(\omega^{(j-1)}\middle|\iota^{(j-1)},\nu^{(j-1)},\underline{x}\right)}\right\}.$$

5. Generate samples U_i, i =1, 2, 3 from the uniform U (0,1) distribution.
6. If $U_1 \le \Theta_{\iota}, U_2 \le \Theta_{\nu}$, and $U_3 \le \Theta_{\omega}$, then set $\iota^{(j)}=\iota^{*}$, $\nu^{(j)}=\nu^{*}$, $\omega^{(j)}=\omega^{*}$; otherwise $\iota^{(j)}=\iota^{(j-1)}$, $\nu^{(j)}=\nu^{(j-1)}$, $\omega^{(j)}=\omega^{(j-1)}$.
7. Set $j=j+1$.
8. Repeat steps 3–7 N times and obtain $\iota^{(j)}$, $\nu^{(j)}$, and $\omega^{(j)}$, for $j=1,2,....N$.

4.3 Simulation Study

In this subsection, a simulation study is used to determine the efficiency of the ML estimates (MLEs) and Bayesian estimates (BEs). This is done by generating 1000 samples from the SIEW distribution using three alternative sample sizes: 75, 100, and 200. Analysis of BEs with SELF, LINEX, and MLF LFs using MCMC is done by focusing on cases where $c = -2$ in Cases 1 and 2, and $c = 2$ in Cases 3 and 4. The mean, bias and mean squared error (MSE) are computed. The results of the simulation study for the SIEW distribution are presented in Tables 3 to 6, which discuss various scenarios.

Case 1: set 1= $(\iota=1.5, \nu=0.5, \omega=0.1)$, set 2= $(\iota=2, \nu=0.5, \omega=0.1)$, and set 3= $(\iota=2.5, \nu=0.5, \omega=0.1)$;

Case 2: set 4= $(\iota=1.5, \nu=0.9, \omega=0.1)$, set 5= $(\iota=2, \nu=0.9, \omega=0.1)$, and set 6= $(\iota=2.5, \nu=0.9, \omega=0.1)$;

Case 3: set 1= $(\iota=1.5, \nu=0.5, \omega=0.1)$, set 2= $(\iota=2, \nu=0.5, \omega=0.1)$, and set 3= $(\iota=2.5, \nu=0.5, \omega=0.1)$;

Case 4: set 4= $(\iota=1.5, \nu=0.9, \omega=0.1)$, set 5= $(\iota=2, \nu=0.9, \omega=0.1)$, and set 6= $(\iota=2.5, \nu=0.9, \omega=0.1)$.

Here, M-H algorithm Metropolis et al. (1953) and Hastings (1970) will be used via R 4.3.2 program. For the ML estimation, the HD Interval package HD Interval is utilized, with the following assumptions,10000 iterations are required for the estimation procedure; the first 20% are discarded as a burn-in sample. The estimate is found to work best when all hyper-parameters have values larger than one and correspond to a unimodal gamma PDF. The outcomes repeatedly show that the Bayesian and ML approaches are beneficial. As the sample size increases, bias and MSE decrease. Furthermore, it's observed that the BE outperforms the MLE in practically all cases. Generally, the results show the reliability and consistency of the estimating methods employed in present work. The employed methodologies demonstrated that they work well under a variety of conditions after effectively estimating SIEW distribution parameters.

In the provided simulation, the MSE findings are shown in Tables 3–6. For both MLE and BE, all estimation methods work perfectly with little bias and low MSE for both BE and MLE. The estimates' mean values are very close to the parameters' real values. Our observations show that BE performs better than MLE in every case. We have seen that BE clearly outperforms other estimation techniques when an asymmetric LF is used.

Table 3. Mean, Bias, MSE and for MLE and BE: Case 1

n	Estimate	Measures	set 1			set 2			set 3		
			$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$
75	MLE	mean	1.712897	0.608469	0.153055	2.234204	0.618486	0.158391	2.888015	0.597200	0.145412
		Bias	0.212897	0.108469	0.053055	0.234204	0.118486	0.058391	0.388015	0.097200	0.045412
		MSE	0.592542	0.278637	0.037332	0.890535	0.270974	0.041665	1.622691	0.319342	0.037245
	SELF	mean	1.712907	0.608366	0.153069	2.234259	0.618419	0.158314	2.887900	0.597104	0.145197
		Bias	0.212907	0.108366	0.053069	0.234259	0.118419	0.058314	0.387900	0.097104	0.045197
		MSE	0.045334	0.011748	0.002821	0.054882	0.014027	0.003405	0.150513	0.009476	0.002090
	LINEX	mean	1.712909	0.608367	0.153073	2.23426	0.61842	0.158339	2.887913	0.597117	0.145334
		Bias	0.212909	0.108367	0.053073	0.23426	0.11842	0.058339	0.387913	0.097117	0.045334
		MSE	0.045335	0.011748	0.002819	0.054883	0.014028	0.003405	0.150523	0.009478	0.002075
	MLF	mean	1.712906	0.608361	0.153102	2.234258	0.618414	0.15839	2.887891	0.597059	0.145752
		Bias	0.212906	0.108361	0.053102	0.234258	0.118414	0.05839	0.387891	0.097059	0.045752
		MSE	0.045334	0.011747	0.002825	0.054882	0.014026	0.113707	0.150506	0.009467	0.002141
100	MLE	mean	1.653769	0.589581	0.144487	2.200712	0.579427	0.143395	2.778054	0.555163	0.132324
		Bias	0.153769	0.089581	0.044487	0.200712	0.079427	0.043395	0.278054	0.055163	0.032324
		MSE	0.402583	0.221857	0.029765	0.699738	0.133278	0.024929	0.996095	0.115609	0.020686
	SELF	mean	1.653757	0.589532	0.144424	2.200754	0.579506	0.143384	2.778039	0.55513	0.132259
		Bias	0.153757	0.089532	0.044424	0.200754	0.079506	0.043384	0.278039	0.05513	0.032259
		MSE	0.023646	0.008021	0.001978	0.040307	0.006326	0.001887	0.077311	0.003044	0.001045
	LINEX	mean	1.653758	0.589533	0.144528	2.200756	0.579507	0.143332	2.77804	0.555131	0.13232
		Bias	0.153758	0.089533	0.044528	0.200756	0.079507	0.043332	0.27804	0.055131	0.03232
		MSE	0.023646	0.008021	0.001985	0.040307	0.006326	0.00188	0.077311	0.003044	0.001046
	MLF	mean	1.653755	0.589527	0.14466	2.200753	0.579501	0.143309	2.778038	0.555125	0.132413
		Bias	0.153755	0.089527	0.04466	0.200753	0.079501	0.043309	0.278038	0.055125	0.032413
		MSE	0.023645	0.00802	0.001999	0.040306	0.006325	0.001881	0.07731	0.003043	0.001055
200	MLE	mean	1.572744	0.532976	0.118864	2.12563	0.514381	0.11005	2.623815	0.524976	0.114392
		Bias	0.072744	0.032976	0.018864	0.12563	0.014381	0.01005	0.123815	0.024976	0.014392
		MSE	0.135862	0.047547	0.009661	0.230493	0.03542	0.007148	0.336941	0.037248	0.007525
	SELF	mean	1.572775	0.532967	0.11871	2.12546	0.514356	0.109946	2.623771	0.524854	0.114352
		Bias	0.072775	0.032967	0.01871	0.12546	0.014356	0.009946	0.123771	0.024854	0.014352
		MSE	0.005301	0.001092	0.000355	0.015745	0.000211	0.000103	0.015324	0.000622	0.00021
	LINEX	mean	1.572776	0.532969	0.118803	2.125461	0.514358	0.10999	2.623772	0.524855	0.114365
		Bias	0.072776	0.032969	0.018803	0.125461	0.014358	0.00999	0.123772	0.024855	0.014365
		MSE	0.005301	0.001092	0.000356	0.015745	0.000211	0.000102	0.015324	0.000623	0.000208
	MLF	mean	1.572773	0.532962	0.11893	2.125459	0.514351	0.110068	2.62377	0.524849	0.114412
		Bias	0.072773	0.032962	0.01893	0.125459	0.014351	0.010068	0.12377	0.024849	0.014412
		MSE	0.005301	0.001091	0.000363	0.015744	0.00021	0.000106	0.015324	0.000622	0.000212

Table 4. Mean, Bias, MSE and for MLE and BE: Case 2

n	Estimate	Measures	set 4			set 5			set 6		
			$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$
75	MLE	mean	1.668647	1.234352	0.161041	2.236464	1.192305	0.155998	2.730807	1.238434	0.164612
		Bias	0.168647	0.334352	0.061041	0.236464	0.292305	0.055998	0.230807	0.338434	0.064612
		MSE	0.430643	2.233002	0.048158	0.768388	1.704470	0.042345	1.108166	1.796411	0.043903
	SELF	mean	1.662186	1.235257	0.138028	2.231080	1.192986	0.132772	2.726169	1.238927	0.141110
		Bias	0.162186	0.335257	0.038028	0.231080	0.292986	0.032772	0.226169	0.338927	0.041110
		MSE	0.026307	0.112399	0.001448	0.053400	0.085843	0.001076	0.051155	0.114874	0.001692
	LINEX	mean	1.662192	1.235257	0.153426	2.231085	1.192987	0.148297	2.726172	1.238928	0.156815
		Bias	0.162192	0.335257	0.053426	0.231085	0.292987	0.048297	0.226172	0.338928	0.056815
		MSE	0.026309	0.112400	0.002855	0.053402	0.085843	0.002333	0.051156	0.114874	0.003229
	MLF	mean	1.662178	1.235255	0.171273	2.231076	1.192985	0.166419	2.726166	1.238926	0.175009
		Bias	0.162178	0.335255	0.071273	0.231076	0.292985	0.066419	0.226166	0.338926	0.075009
		MSE	0.026304	0.112398	0.005082	0.053398	0.085842	0.004414	0.051153	0.114873	0.005629
100	MLE	mean	1.646780	1.091878	0.139742	2.150014	1.151918	0.148815	2.655176	1.137725	0.150558
		Bias	0.146780	0.191878	0.039742	0.150014	0.251918	0.048815	0.155176	0.237725	0.050558
		MSE	0.307034	0.820629	0.029133	0.529989	1.442536	0.030957	0.719740	0.751912	0.027128
	SELF	mean	1.640231	1.092449	0.116810	2.144583	1.152434	0.125535	2.651070	1.138346	0.126980
		Bias	0.140231	0.192449	0.016810	0.144583	0.252434	0.025535	0.151070	0.238346	0.026980
		MSE	0.019667	0.037039	0.000284	0.020907	0.063725	0.000654	0.022825	0.056811	0.000729
	LINEX	mean	1.640237	1.092449	0.132179	2.144588	1.152434	0.141106	2.651073	1.138347	0.142730
		Bias	0.140237	0.192449	0.032179	0.144588	0.252434	0.041106	0.151073	0.238347	0.042730
		MSE	0.019669	0.037039	0.001036	0.020908	0.063725	0.001690	0.022826	0.056811	0.001827
	MLF	mean	1.640222	1.092447	0.150465	2.144578	1.152432	0.159452	2.651068	1.138345	0.161276
		Bias	0.140222	0.192447	0.050465	0.144578	0.252432	0.059452	0.151068	0.238345	0.061276
		MSE	0.019665	0.037038	0.002549	0.020905	0.063724	0.003537	0.022824	0.056811	0.003757
200	MLE	mean	1.555764	0.981040	0.120087	2.060559	0.993284	0.122981	2.599895	0.973427	0.118172
		Bias	0.055764	0.081040	0.020087	0.060559	0.093284	0.022981	0.099895	0.073427	0.018172
		MSE	0.104316	0.207365	0.009815	0.189440	0.198054	0.009673	0.310218	0.176868	0.008392
	SELF	mean	1.547606	0.978215	0.101052	2.054238	0.991605	0.101740	2.592484	0.968080	0.100254
		Bias	0.047606	0.078215	0.001052	0.054238	0.091605	0.001740	0.092484	0.068080	0.000254
		MSE	0.002269	0.006120	0.000001	0.002944	0.008394	0.000004	0.008556	0.004637	0.000043
	LINEX	mean	1.547629	0.978229	0.112400	2.054253	0.991613	0.115170	2.592506	0.968107	0.110325
		Bias	0.047629	0.078229	0.012400	0.054253	0.091613	0.015170	0.092506	0.068107	0.010325
		MSE	0.002271	0.006122	0.000154	0.002946	0.008395	0.000231	0.008560	0.004641	0.000107
	MLF	mean	1.547576	0.978186	0.125305	2.054223	0.991589	0.130911	2.592467	0.968022	0.121590
		Bias	0.047576	0.078186	0.025305	0.054223	0.091589	0.030911	0.092467	0.068022	0.021590
		MSE	0.002266	0.006116	0.000643	0.002943	0.008391	0.000958	0.008553	0.004629	0.000468

Table 5. Mean, Bias, MSE and for MLE and BE: Case 3

n	Estimate	Measures	set 1			set 2			set 3		
			$\hat{\iota}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\iota}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\iota}$	$\hat{\nu}$	$\hat{\omega}$
75	MLE	mean	1.747767	0.594233	0.147260	2.312818	0.582060	0.141609	2.860336	0.622201	0.155756
		Bias	0.247767	0.094233	0.047260	0.312818	0.082060	0.041609	0.360336	0.122201	0.055756
		MSE	0.615099	0.222441	0.035986	0.986966	0.228211	0.031071	1.716841	0.414900	0.042644
	SELF	mean	1.747872	0.594293	0.147145	2.312676	0.582040	0.141687	2.860265	0.622317	0.155550
		Bias	0.247872	0.094293	0.047145	0.312676	0.082040	0.041687	0.360265	0.122317	0.055550
		MSE	0.061445	0.008896	0.002227	0.097771	0.006735	0.001743	0.129838	0.015008	0.003133
	LINEX	mean	1.747870	0.594292	0.147163	2.312675	0.582039	0.141705	2.860252	0.622304	0.155670
		Bias	0.247870	0.094292	0.047163	0.312675	0.082039	0.041705	0.360252	0.122304	0.055670
		MSE	0.061444	0.008896	0.002226	0.097770	0.006735	0.001741	0.129828	0.015005	0.003120
	MLF	mean	1.747870	0.594289	0.147208	2.312675	0.582036	0.141754	2.860256	0.622275	0.156049
		Bias	0.247870	0.094289	0.047208	0.312675	0.082036	0.041754	0.360256	0.122275	0.056049
		MSE	0.061444	0.008895	0.002233	0.097770	0.006734	0.001748	0.129831	0.014997	0.003189
100	MLE	mean	1.619090	0.592640	0.143477	2.245908	0.574818	0.133247	2.763566	0.547279	0.129986
		Bias	0.119090	0.092640	0.043477	0.245908	0.074818	0.033247	0.263566	0.047279	0.029986
		MSE	0.317066	0.196783	0.025675	0.702086	0.487363	0.027969	0.987938	0.079343	0.016795
	SELF	mean	1.619072	0.592606	0.143438	2.245923	0.574837	0.133048	2.763664	0.547248	0.130005
		Bias	0.119072	0.092606	0.043438	0.245923	0.074837	0.033048	0.263664	0.047248	0.030005
		MSE	0.014183	0.008581	0.001892	0.060483	0.005605	0.001097	0.069523	0.002237	0.000905
	LINEX	mean	1.619071	0.592604	0.143467	2.245922	0.574836	0.133146	2.763662	0.547247	0.129963
		Bias	0.119071	0.092604	0.043467	0.245922	0.074836	0.033146	0.263662	0.047247	0.029963
		MSE	0.014183	0.008580	0.001892	0.060482	0.005605	0.001101	0.069522	0.002237	0.000900
	MLF	mean	1.619070	0.592601	0.143525	2.245922	0.574832	0.133273	2.763663	0.547243	0.129953
		Bias	0.119070	0.092601	0.043525	0.245922	0.074832	0.033273	0.263663	0.047243	0.029953
		MSE	0.014183	0.008580	0.001900	0.060482	0.005604	0.001111	0.069523	0.002237	0.000902
200	MLE	mean	1.567529	0.528422	0.117608	2.092554	0.528085	0.116320	2.633460	0.520263	0.113250
		Bias	0.067529	0.028422	0.017608	0.092554	0.028085	0.016320	0.133460	0.020263	0.013250
		MSE	0.132337	0.036785	0.007830	0.229411	0.040010	0.008056	0.341635	0.037218	0.007454
	SELF	mean	1.567513	0.528402	0.117595	2.092555	0.528114	0.116243	2.633507	0.520309	0.113280
		Bias	0.067513	0.028402	0.017595	0.092555	0.028114	0.016243	0.133507	0.020309	0.013280
		MSE	0.004563	0.000811	0.000314	0.008571	0.000795	0.000269	0.017829	0.000417	0.000181
	LINEX	mean	1.567511	0.528400	0.117596	2.092554	0.528112	0.116304	2.633506	0.520308	0.113254
		Bias	0.067511	0.028400	0.017596	0.092554	0.028112	0.016304	0.133506	0.020308	0.013254
		MSE	0.004562	0.000811	0.000312	0.008571	0.000795	0.000268	0.017828	0.000417	0.000178
	MLF	mean	1.567511	0.528397	0.117632	2.092554	0.528109	0.116398	2.633506	0.520304	0.113263
		Bias	0.067511	0.028397	0.017632	0.092554	0.028109	0.016398	0.133506	0.020304	0.013263
		MSE	0.004562	0.000811	0.000316	0.008571	0.000795	0.000273	0.017828	0.000417	0.000180

Table 6. Mean, Bias, MSE and for MLE and BE: Case 4.

n	Estimate	Measures	set 4			set 5			set 6		
			$\hat{\iota}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\iota}$	$\hat{\nu}$	$\hat{\omega}$	$\hat{\iota}$	$\hat{\nu}$	$\hat{\omega}$
75	MLE	mean	1.644875	1.280096	0.170414	2.209666	1.229805	0.164007	2.743829	1.084620	0.139775
		Bias	0.144875	0.380096	0.070414	0.209666	0.329805	0.064007	0.243829	0.184620	0.039775
		MSE	0.471603	2.199931	0.046782	0.754896	1.597333	0.045392	0.872331	0.903428	0.028012
	SELF	mean	1.637259	1.280331	0.149192	2.203743	1.230109	0.140849	2.740150	1.085402	0.116140
		Bias	0.137259	0.380331	0.049192	0.203743	0.330109	0.040849	0.240150	0.185402	0.016140
		MSE	0.018843	0.144655	0.002423	0.041514	0.108974	0.001670	0.057674	0.034376	0.000262
	LINEX	mean	1.637249	1.280331	0.164195	2.203738	1.230108	0.156304	2.740147	1.085401	0.131915
		Bias	0.137249	0.380331	0.064195	0.203738	0.330108	0.056304	0.240147	0.185401	0.031915
		MSE	0.018840	0.144654	0.004123	0.041511	0.108974	0.003171	0.057673	0.034376	0.001019
	MLF	mean	1.637247	1.280330	0.181338	2.203738	1.230108	0.174174	2.740148	1.085400	0.150790
		Bias	0.137247	0.380330	0.081338	0.203738	0.330108	0.074174	0.240148	0.185400	0.050790
		MSE	0.018839	0.144654	0.006619	0.041511	0.108974	0.005504	0.057673	0.034376	0.002582
100	MLE	mean	1.616424	1.154162	0.148423	2.139181	1.115369	0.143812	2.749906	1.225878	0.161110
		Bias	0.116424	0.254162	0.048423	0.139181	0.215369	0.043812	0.249906	0.325878	0.061110
		MSE	0.305592	1.921994	0.032414	0.463318	0.851362	0.026243	1.140921	1.915261	0.043287
	SELF	mean	1.609634	1.154640	0.125594	2.134174	1.115879	0.120405	2.745913	1.226665	0.137526
		Bias	0.109634	0.254640	0.025594	0.134174	0.215879	0.020405	0.245913	0.326665	0.037526
		MSE	0.012022	0.064844	0.000657	0.018005	0.046606	0.000418	0.060475	0.106713	0.001410
	LINEX	mean	1.609626	1.154640	0.140852	2.134170	1.115879	0.136057	2.745910	1.226664	0.153253
		Bias	0.109626	0.254640	0.040852	0.134170	0.215879	0.036057	0.245910	0.326664	0.053253
		MSE	0.012020	0.064844	0.001670	0.018004	0.046606	0.001301	0.060474	0.106712	0.002837
	MLF	mean	1.609624	1.154639	0.158762	2.134170	1.115878	0.154635	2.745911	1.226664	0.171548
		Bias	0.109624	0.254639	0.058762	0.134170	0.215878	0.054635	0.245911	0.326664	0.071548
		MSE	0.012020	0.064843	0.003455	0.018004	0.046606	0.002987	0.060474	0.106712	0.005122
200	MLE	mean	1.549221	0.976467	0.119797	2.079046	0.989176	0.121016	2.582741	0.985644	0.121071
		Bias	0.049221	0.076467	0.019797	0.079046	0.089176	0.021016	0.082741	0.085644	0.021071
		MSE	0.103251	0.168956	0.008471	0.206444	0.206842	0.009647	0.281664	0.201447	0.009533
	SELF	mean	1.542860	0.975394	0.099153	2.072950	0.987715	0.099704	2.576525	0.982696	0.101777
		Bias	0.042860	0.075394	0.000847	0.072950	0.087715	0.000296	0.076525	0.082696	0.001777
		MSE	0.001839	0.005687	0.000021	0.005324	0.007696	0.000031	0.005858	0.006841	0.000003
	LINEX	mean	1.542847	0.975386	0.112064	2.072935	0.987708	0.113178	2.576506	0.982679	0.113223
		Bias	0.042847	0.075386	0.012064	0.072935	0.087708	0.013178	0.076506	0.082679	0.013223
		MSE	0.001838	0.005686	0.000146	0.005322	0.007695	0.000174	0.005856	0.006838	0.000176
	MLF	mean	1.542843	0.975378	0.127120	2.072936	0.987700	0.129021	2.576510	0.982662	0.126235
		Bias	0.042843	0.075378	0.027120	0.072936	0.087700	0.029021	0.076510	0.082662	0.026235
		MSE	0.001838	0.005684	0.000738	0.005322	0.007694	0.000844	0.005856	0.006835	0.000691

Figures 2 and 3 present convergence plots of MCMC for parameter estimates of the SIEW distribution in set 3 with sample size (n = 75), employing both symmetric and asymmetric LFs across 10000 iterations. The findings demonstrate a strong resemblance between the estimates and the theoretical posterior density functions. It is quite likely that more rounds of the MCMC approach would yield similar and even better results, as suggested.

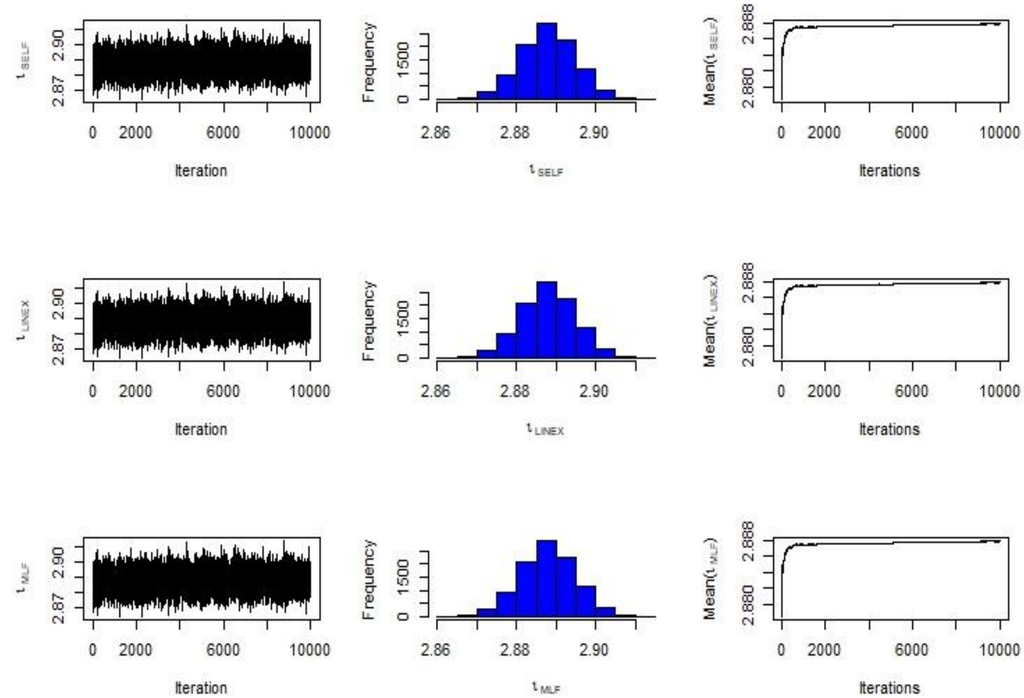

Figure 2: The MCMC plots of ι for the SIEW distribution in Case 1.

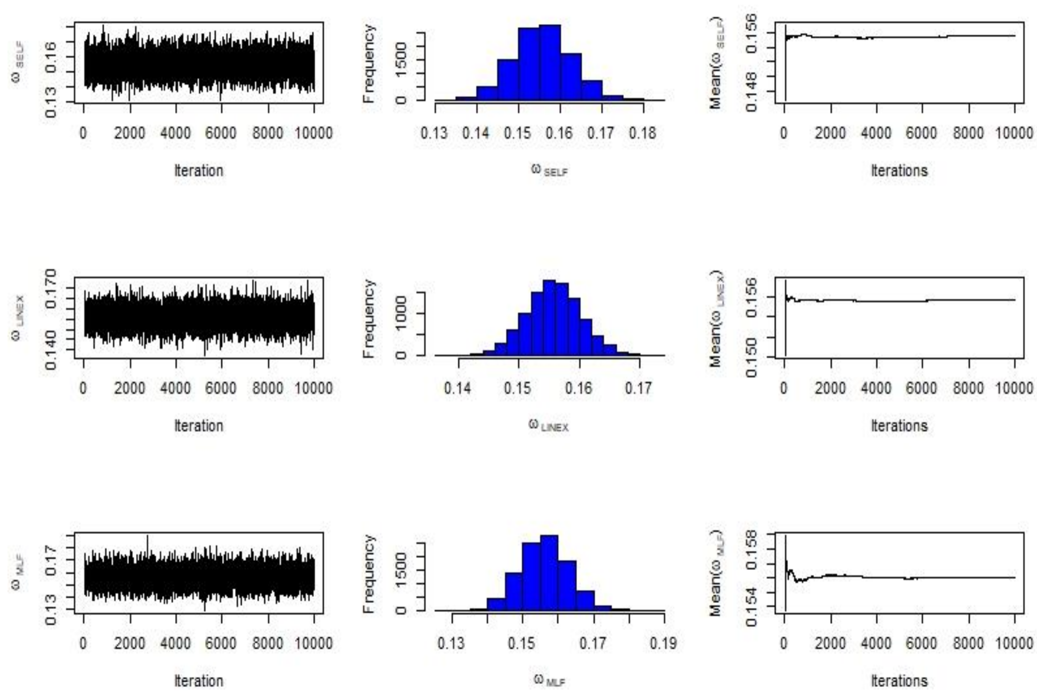

Figure 3. The MCMC plots of ω for the SIEW distribution in Case 3.

As shown in Figures 2 and 3, the algorithm performs well with the given initial condition and proposal distribution. Although the samples are correlated, the Markov Chain mixes well, and the trace within any iteration remains consistent, indicating rapid convergence in distribution. The MCMC plots confirm that the posterior distributions of the parameters follow a normal distribution, and the averaged estimates of ι and ω under both symmetric and asymmetric LFs converge after 2000 iterations.

5. Application

In this section of our research, two applications are used to show the flexibility and efficiency of the SIEW distribution. Once we focus on modeling tools, the SIEW distribution is compared to other competing distributions, including IEW distribution, log-normal (log-N) distribution (Torrent (1978)), Weibull exponential (WE) distribution (Bilal *et al.* (2021)), exponential Pareto (EP) distribution (Al-Kadim and Boshi (2013)), Burr XII distribution (Zimmer *et al.* (1998)), and transmuted power inverted Topp–Leone (TPITL) distribution (Nassr *et al.* (2022)).

Different models are evaluated and compared using nine well-referenced measures of goodness-of-fit. These measures include minus two log-likelihood ($\Im_0$), Akaike information criterion ($\Im_1$), Bayesian information criterion ($\Im_2$), Akaike information criterion corrected ($\Im_3$), Hannan-Quinn information criterion ($\Im_4$), Kolmogorov-Smirnov test ($\Im_5$), Anderson-Darling statistics (A^*), and Cramér-von Mises statistic (W^*), (see Alomani et al. (2024)). The model that exhibits the lowest values for these statistics and measures is considered the best fit. Additionally, we analyze the P-values ($\Im_6$) associated with the $\Im_5$. A model with the highest $\Im_6$ value is considered the best among the tested models.

5.1 Failure Time Data

Murthy et al. (2004) presented the following data on Failure interval time for 30 fixed components:

1.43	0.11	0.71	0.77	2.63	1.49	3.46	2.46	0.59	0.74
1.23	0.94	4.36	0.4	1.74	4.73	2.23	2.23	0.7	1.06
1.46	0.3	1.82	2.37	0.63	1.23	1.24	1.24	1.186	1.17

Table 7 presents the corresponding summary statistics of the above dataset. Table 8 presents the MLEs for this dataset, accompanied by their standard errors (SEs). In Table 9, we can observe the numerical values of several statistical measures, such as $\Im_0$, $\Im_1$, $\Im_2$, $\Im_3$, $\Im_4$, $\Im_5$, $\Im_6$, A^*, and W^*. Figure 4 displays the estimation of the first PDF form using an informal density method. It is evident that the PDF shape exhibits asymmetrical. Furthermore, the figure includes a quantile-quantile (QQ) plot, which is employed to assess the normality assumption. By employing a box plot, outliers within the first dataset can be identified. Consequently, our analysis indicates the presence of outliers in the dataset, with the red mark denoting the mean and the blue rings describing data sets. Figure 5 shows cases of the empirical PDF (EPDF) and empirical CDF (ECDF) for the different models. Comparing the competing models on the first dataset, Figure 6 shows probability-probability (PP) plots. The good fit of our model to the data is shown in Figures 5 and 6.

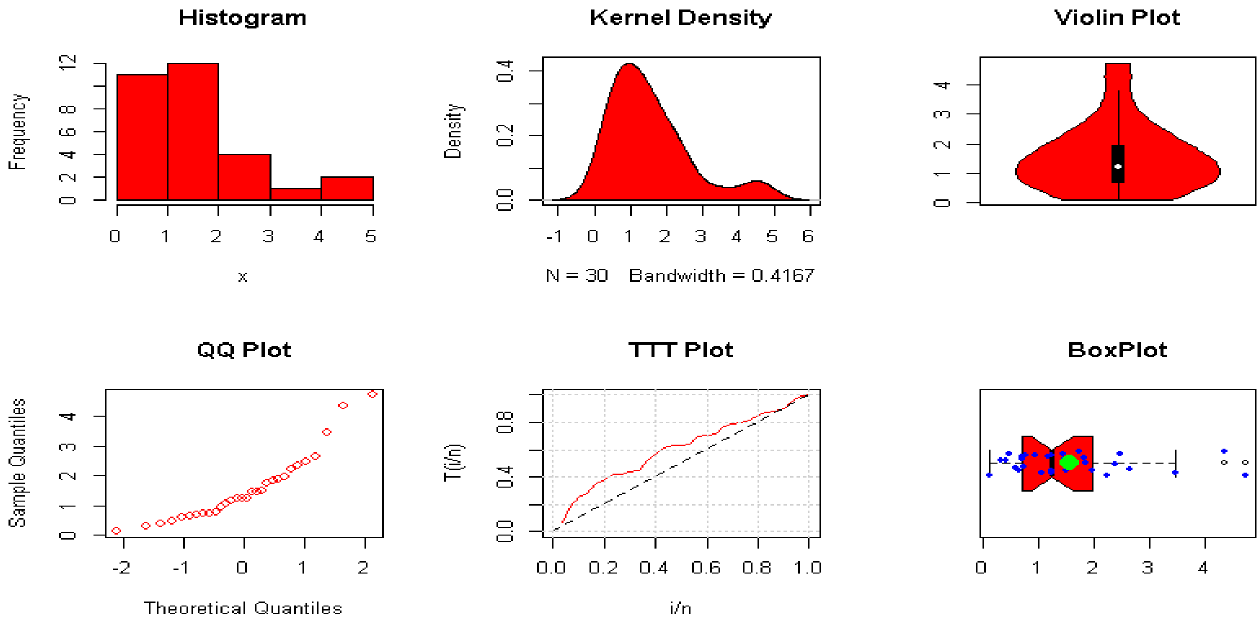

Figure 4. A few simple informal graphs of times between failures charts

Table 7. Summary statistics for time between failures.

Minimum	Q_1	Median	Mean	Q_3	Maximum	σ^2	SK	KU
0.110	0.718	1.235	1.543	1.943	4.730	1.272	1.296	4.319

Table 8. MLEs and SEs of the competitive distributions for time between failures.

Distribution	$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$	SE ($\hat{\imath}$)	SE ($\hat{\nu}$)	SE ($\hat{\omega}$)
SIEW	0.294628	48.9093	5.119252	0.098816	76.38914	1.611127
IEW	0.332248	57.12423	4.705269	0.130678	101.9672	1.848326
log-N	0.159692	0.801887	-	0.146404	0.103525	-
WE	32.40809	1.395396	0.045801	48.45745	0.189309	0.043361
EP	1.735632	1.463319	1.022025	117.9913	0.202908	101.67
Burr XII	2.370873	0.80984	-	0.421826	0.172517	-
TPITL	-0.62845	1.17488	2.817191	0.525116	0.335284	1.051332

Table 9. Measures of fitting for time between failures.

Distribution	$\mathfrak{I}_0$	$\mathfrak{I}_1$	$\mathfrak{I}_2$	$\mathfrak{I}_3$	$\mathfrak{I}_4$	$\mathfrak{I}_5$	$\mathfrak{I}_6$	W^*	A^*
SIEW	39.796	85.592	89.295	86.515	86.736	0.067	0.999	0.020	0.146
IEW	40.014	86.028	90.232	86.951	87.373	0.083	0.985	0.024	0.174
log-N	40.735	85.670	89.673	86.915	86.367	0.099	0.932	0.039	0.274
WE	40.026	86.052	90.256	86.975	87.397	0.081	0.989	0.032	0.238
EP	39.910	85.821	90.024	86.744	87.166	0.075	0.996	0.028	0.212
Burr XII	40.906	85.812	89.615	86.957	86.909	0.116	0.813	0.043	0.273
TPITL	40.108	86.216	90.420	87.139	87.561	0.085	0.982	0.020	0.211

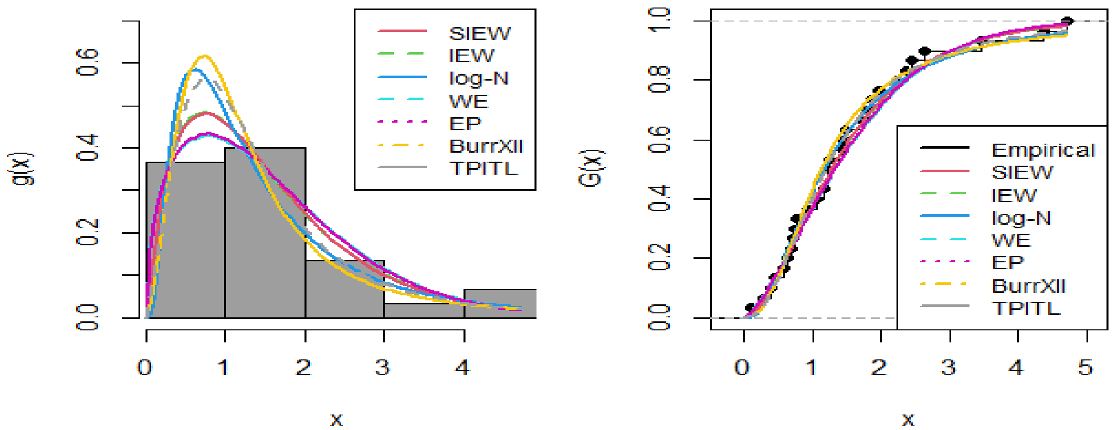

Figure 5. Plots of estimated EPDF and ECDF for time between failures

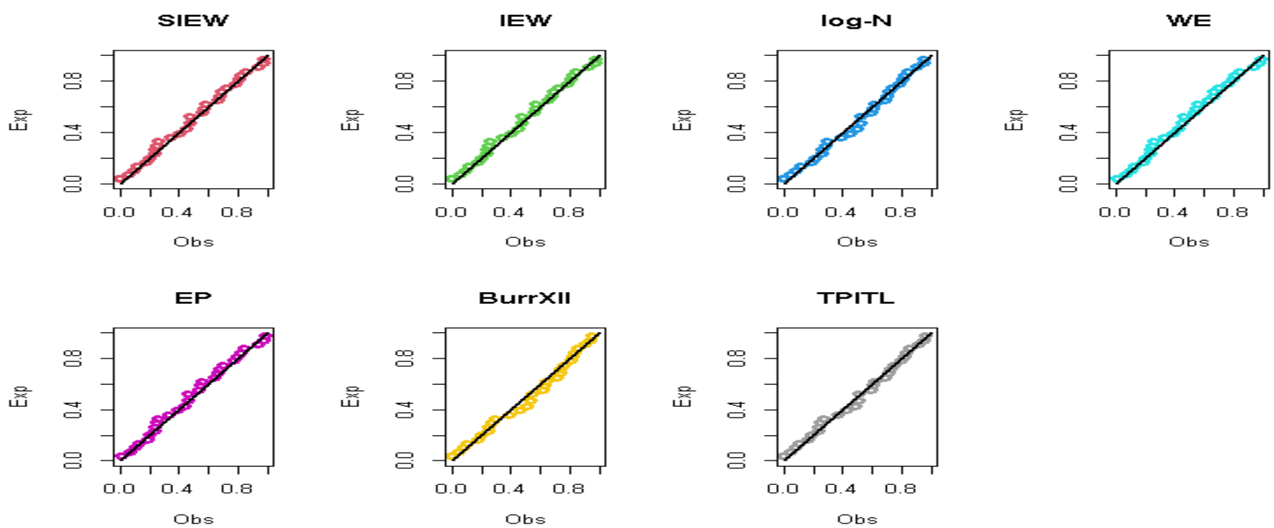

Figure 6. PP plots for the fitted distributions for the time between failures

5.2 Fatigue Fracture Data

The lifetime of Kevlar 373/epoxy fatigue fractures under continuous stress, applied at 90% of maximum tension, is investigated using 76 observations from a dataset discussed by Owoloko *et al.* (2015). A description of this data is given below, along with the non-parametric graphs in Figure 7 and the descriptive statistics for the second data set displayed in Table 10.

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.565	0.5671	0.6566
0.6748	0.6751	0.6752	0.7696	0.8375	0.8391	0.8425	0.8645	0.5581	0.9113
0.912	0.9836	1.0483	1.0596	1.0773	1.1733	1.257	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.488	1.5728	1.5733	1.7083	1.7263	1.746	1.763
1.7746	1.8275	1.8375	1.8503	1.8808	1.8878	1.8881	1.9316	1.9558	2.0048
2.0408	2.0903	2.1093	2.133	2.21	2.246	2.2878	2.3203	2.347	2.3513
2.4951	2.526	2.9911	3.0256	3.2678	3.4045	3.4846	3.7433	3.7455	3.9142
4.8073	5.4005	5.4435	5.5295	6.5541	9.096				

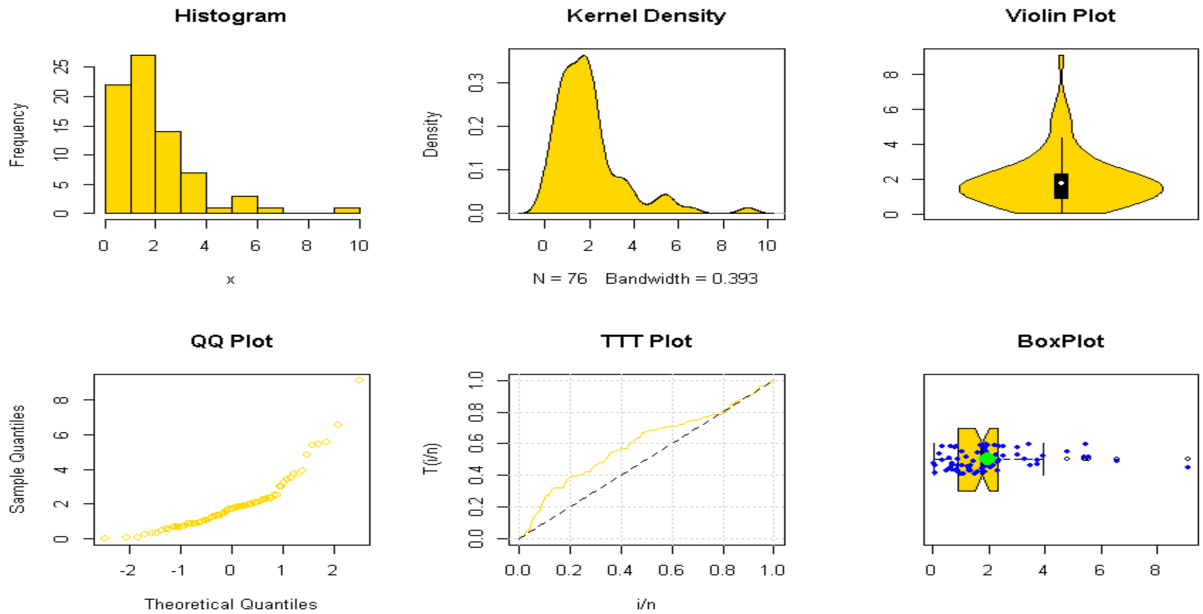

Figure 7. A few simple informal graphs of Fatigue Fracture

Table 10. Summary statistics for Fatigue Fracture

Minimum	Q_1	Median	Mean	Q_3	Maximum	σ^2	SK	KU
0.0251	0.9048	1.7362	1.9592	2.2959	9.0960	2.4774	1.9796	8.1608

Figure 7 shows a visual representation of the density estimate process using an informal graphical technique for calculating the PDF of the fatigue fracture dataset. In addition, the QQ plot in the same figure is used to evaluate the normality conclusion and a box plot is used to identify outliers. As a result, it is possible to determine that outliers exist in the Fatigue Fracture dataset (the mark with the green color represents the median, but the blue rings describe data sets). Table 11 shows the MLEs for this dataset, along with their SEs. Additionally, Table 12 displays the numerical values for several statistical indicators derived from the fatigue fracture data, including $\Im_0$, $\Im_1$, $\Im_2$, $\Im_3$, $\Im_4$, $\Im_5$, $\Im_6$, A^*, and W^*. Figure 8 displays the PDFs and CDFs of the competing models, while Figure 9 presents the PP plots of these models on the second dataset. Our model fits the data well, as shown by the graphs in Figures 8 and 9.

Table 11. MLEs and SEs of the competitive distributions for Fatigue Fracture Data

Distribution	$\hat{\imath}$	$\hat{\nu}$	$\hat{\omega}$	SE ($\hat{\imath}$)	SE ($\hat{\nu}$)	SE ($\hat{\omega}$)
SIEW	0.1202	15027	11.098	0.0175	18840	1.2702
IEW	0.1542	3940.2	9.2319	0.0341	7169.1	1.845
log-N	0.3382	0.9546	-	0.1095	0.0774	-
WE	472.24	1.3196	0.0044	271.07	0.1098	0.0012
EP	0.1535	1.3257	0.0305	0.2769	0.1138	0.0723
Burr XII	2.2309	0.6655	-	0.2671	0.0946	-
TPITL	-0.717	1.0867	2.4413	0.1777	0.1467	0.3951

Table 12. Measures of fitting for Fatigue Fracture

Distribution	$\Im_0$	$\Im_1$	$\Im_2$	$\Im_3$	$\Im_4$	$\Im_5$	$\Im_6$	W*	A*
SIEW	244.954	250.954	257.946	251.287	253.748	0.091	0.531	0.122	0.731
IEW	246.297	252.297	259.289	252.63	255.091	0.094	0.489	0.137	0.822
log-N	259.998	263.998	268.66	264.163	265.861	0.112	0.273	0.305	1.863
WE	245.156	251.156	258.148	251.489	253.95	0.112	0.273	0.132	0.777
EP	245.049	251.049	258.042	251.383	253.844	0.11	0.295	0.131	0.767
Burr XII	257.107	261.107	265.768	261.271	262.97	0.148	0.066	0.272	1.64
TPITL	249.119	255.119	262.112	255.453	257.914	0.11	0.298	0.169	1.014

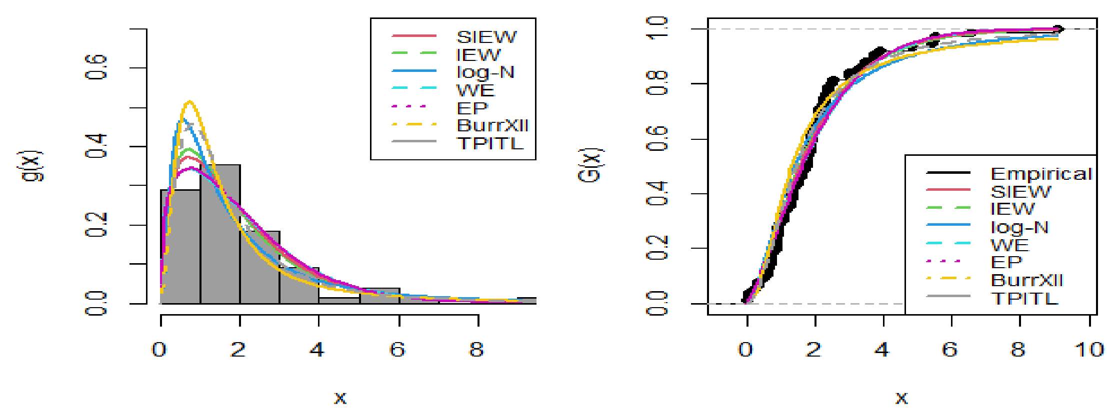

Figure 8. Plots of estimated EPDF and ECDF for Fatigue Fracture

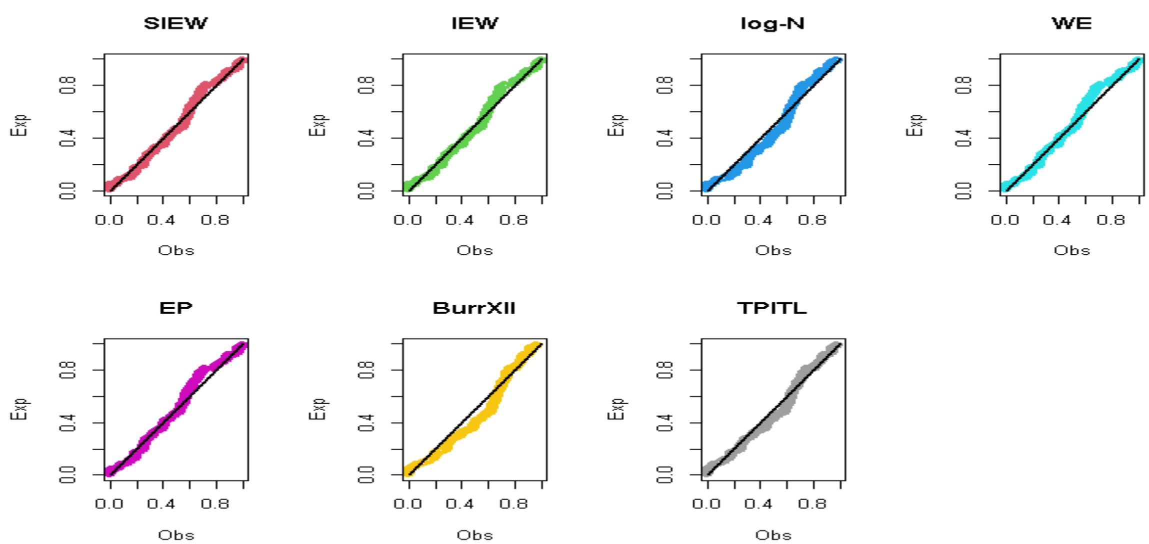

Figure 9. PP plots for the fitted distributions for Fatigue Fracture

6. Conclusion

A new inverted exponentiated Weibull distribution based on the S-G family is presented in this study: the SIEW distribution. With two shape parameters and one scale parameter, the SIEW distribution is flexible and fits a variety of data types. The study investigates several significant SIEW distribution characteristics such as the probability density function, quantile function, hazard rate, moments, incomplete moments, Lorenz and Bonferroni curves, residual and reversed residual moments, stress-strength reliability, and various uncertainty measures. Using the SELF, LINEX, and the MLF LFs, Bayesian technique is used to estimate the distribution's parameters as well as the maximum likelihood estimation method. Monte Carlo simulations are used to analyse the effectiveness of various estimating techniques to compare how well the estimates work. Two actual applications in materials science and engineering—more especially, fracture mechanics—use the SIEW distribution. The SIEW distribution outfits other models on real datasets, as seen by lower criteria measures and greater P-values. This conclusion suggests that the SIEW distribution may produce flexible models with distinctive traits, which limits the possibility of further research in this area.

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Declaration of interests

The authors declare that they have no conflict of interest.

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