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Marshall-Olkin Power Rayleigh Distribution with Properties and Engineering Applications

Yassmen Y. Abdelall *

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Marshall-Olkin, Power Rayleigh distribution, moments, order statistic, maximum likelihood method.

Abstract

This study presents the proposal of a novel three-parameter Rayleigh distribution, namely Marshall-Olkin Power Rayleigh (MOPR) distribution. Marshall-Olkin Rayleigh (MOR), Marshall-Olkin Chi-Square, and Power Rayleigh (PR) are three particular sub-models of the new distribution. Several of its statistical and mathematical characteristics are derived such as explicit moments, mean deviation, quantile function, Rényi entropy measure, order statistics densities and maximum likelihood estimators. The new distribution may be more flexible since the density shapes are symmetrical and left skewed. The reverse hazard function and truncated moments have been used to obtain the characterizations of the suggested distribution. A Monte Carlo simulation has been conducted to assess maximum likelihood estimators' consistency with respect to bias, variance, and mean square error (MSE) measures. In the end, the proposed distribution is applied to an engineering science-related real data sets and it is seen that this distribution is a flexible model that may be a useful alternative to known distributions like Rayleigh, and Power Rayleigh distributions.

1. Introduction

One such approach used by different researchers is power transformation technique by which an extra parameter is added to the parent distribution. Induction of an extra parameter in the parent model usually provides greater flexibility and improves the goodness of fit. There are plethora of researchers who worked on power generalization of probability models, among them are Meniconi and Barry (1996), Ghitany et al. (2013), Zaka and Akhter (2013), Rady et al. (2016), Krishnarani (2016) and Shukla and Shanker (2018).

The Rayleigh distribution is considered to be a useful lifetime distribution. It is very well-known distribution for modeling lifetime data in communication theory, physical science, engineering and medical imaging science. Several generalization of the Rayleigh distribution have been proposed by various authors in recent years among them, Transmuted Rayleigh (2013), Weibull Rayleigh (2015), Odd Generalized Exponential Rayleigh (2016), Weighted Rayleigh (2019), Topp-Leone Rayleigh (2019), Marshall-Olkin Alpha Power Rayleigh (2021) and Topp-Leone Power Rayleigh (2021).

Corresponding author*: yassmenamalek@gmail.com

Department of Mathematical Statistics, Faculty of Graduate Studies and Statistical Research. Cairo University, Egypt

Bhat and Ahmad (2020) was provided by an extended version of the Rayleigh distribution based on power transformation technique, namely PR distribution having two parameters $\alpha > 0$, is a shape parameter and $\theta > 0$, is a scale parameter, with probability density function (pdf) and cumulative distribution function (cdf), respectively given as follows:

$$g(y) = \frac{\alpha}{\theta^2} y^{2\alpha-1} e^{-\left(\frac{y^{2\alpha}}{2\theta^2}\right)} , \quad \alpha, \theta, y > 0, \tag{1}$$

$$G(y) = 1 - e^{-\left(\frac{y^{2\alpha}}{2\theta^2}\right)} , \quad \alpha, \theta, y > 0, \tag{2}$$

The main goal of this paper, we discussed a new generalization of PR distribution called MOPR distribution based on the Marshall-Olkin generated family of distributions. The new distribution has several density shapes based on additional parameter, contains several important distributions as special sub-models, and provides more flexibility in modeling data for real lifetime applications.

The paper is organized as follows. The MOPR distribution and its quantile function are introduced in Section 2. Some statistical functions of the proposed distribution are provided in Section 3, such as the moments, moment generating function, incomplete moments, mean deviations, Rényi entropy, and the density and moments of the order statistics. Section 4 is presented some characterizations of the new distribution. The estimation of the parameters by the maximum likelihood method is investigated in Section 5. A simulation study is performed in Section 6 to show the accuracy of the maximum likelihood parameters estimated. Two applications to Carbon Fibres real data sets are given in Section 7. In Section 8, we offer some concluding remarks.

2. MOPR distribution

The extended Marshall-Olkin generated family was introduced by Marshall and Olkin (1997). For any baseline distribution $G(y)$, the cdf of the family has the form:

$$F(x) = G(y)\left[1 - (1-a)\overline{G}(y)\right]^{-1}, \quad a > 0, \tag{3}$$

where a is additional shape parameter and $\overline{G}(.) = 1 - G(.)$. For PR distribution and Using (2) in (3), we get the cdf of the MOPR distribution with parameter $\underline{\xi} = (a, \alpha, \theta)$ as follows:

$$F_{MOPR}(x, \underline{\xi}) = \left(1 - e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)\left[1 - (1-a) e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right]^{-1}, \; x, a, \alpha, \theta > 0, \tag{4}$$

The associated pdf of new distribution is given by:

$$f_{MOPR}(x, \underline{\xi}) = \left(\frac{a\,\alpha}{\theta^2}\right) x^{2\alpha-1} e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)} \left[1 - (1-a) e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right]^{-2} . \tag{5}$$

Figure 1 illustrates the density shapes of the MOPR distribution are left skewed and symmetrical as the increase of the value of additional parameter a. Figure 2 shows the cumulative shapes of the MOPR distribution is strictly increasing tends to one as the increasing of the value of additional parameter a.

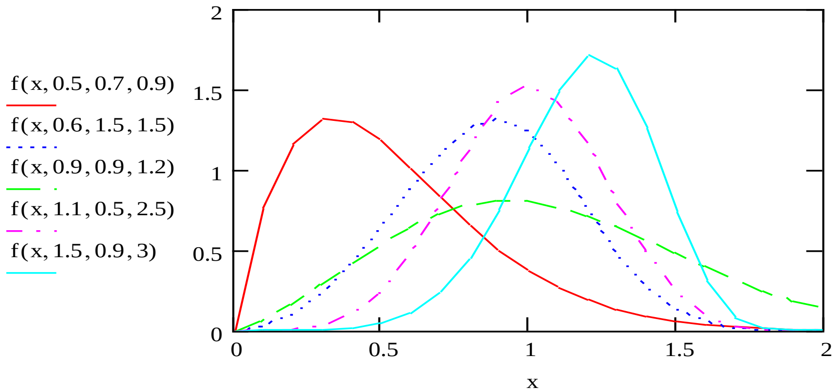

Figure 1: Plots of the pdf of the MOPR distribution.

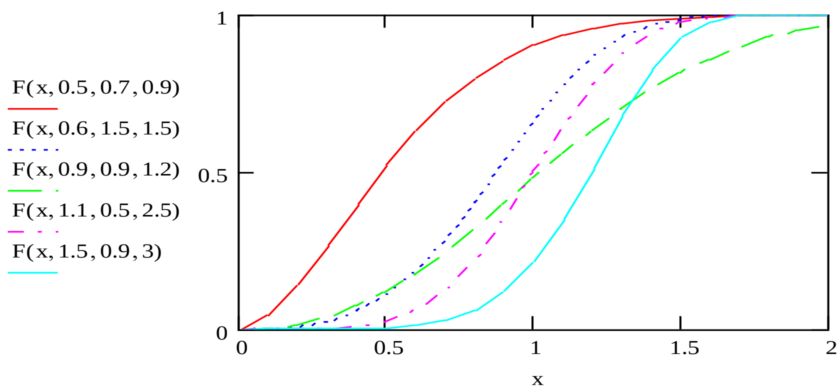

Figure 2: Plots of the cdf of the MOPR distribution.

The corresponding survival, hazard function and reversed hazard functions of MOPR distribution, respectively are given by:

$$S_{MOPR}(x,\underline{\xi}) = 1 - \left(1 - e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)\left[1-(1-a)e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right]^{-1},$$

$$h_{MOPR}(x,\underline{\xi}) = \left(\frac{a\alpha}{\theta^2}\right)x^{2\alpha-1}e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\left[1-(1-a)e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right]^{-1}\left\{\left[1-(1-a)e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right] - \left(1-e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)\right\}^{-1}.$$

and

$$r_{MOPR}(x,\underline{\xi}) = \left(\frac{a\alpha}{\theta^2}\right)x^{2\alpha-1}e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\left(1-e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)^{-1}\left[1-(1-a)\left(e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)\right]^{-1}.$$

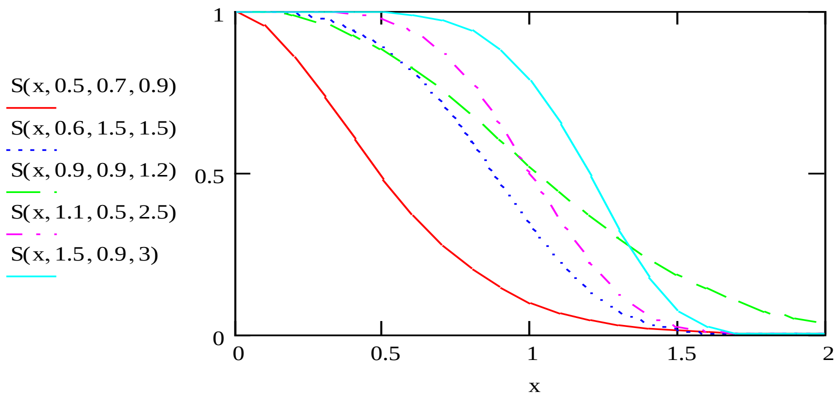

Figure 3: Plot of the survival function of the MOPR distribution

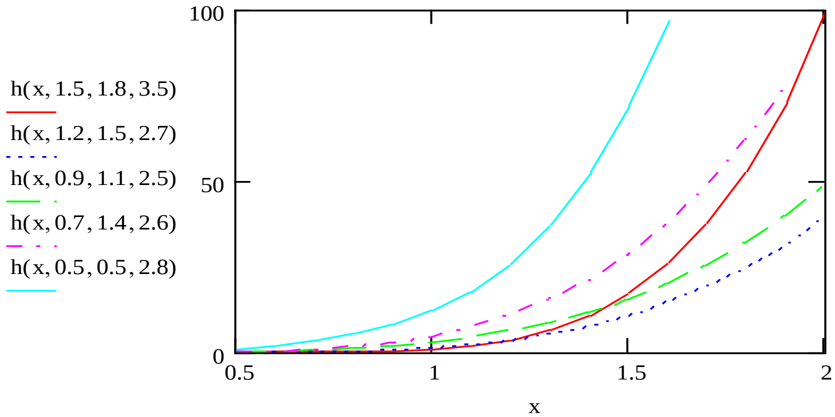

Figure 4: Plot of the hazard function of the MOPR distribution.

The plots of the survival function decreases as the additional parameter a decrease and hazard function is increasing function overall the values of the parameters for the MOPR distribution as illustrated in Figure 3 and 4 respectively.

2.1 Sub models

Different distributions can be deduced of the MOPR distribution when its parameters are changed. The sub-models of X distributed (5) are listed in Table1.

Table1: Sub-models of MOPR distribution

MOPR	**Distribution**
$\alpha = 1$	Marshall-Olkin Rayleigh (MOR) (MirMostafaee et al. (2017))
$\alpha = \theta = 1$	Marshall-Olkin Chi-Square (MO Chi-Square)
$a = 1$	Power Rayleigh (PR) (Bhats and Ahmad (2020))
$\alpha = a = 1$	Rayleigh (R)
$\alpha = \theta = a = 1$	Chi-Square

2.2 Quantile function

The quantile function of the MOPR distribution can be defined as follows:

$$Q(q)=\left[-2\theta^2 \ln\left[1-\frac{q\,a}{1-q(1-a)}\right]\right]^{\frac{1}{2\alpha}}, \quad 0<q<1. \tag{6}$$

First quantile of the MOPR distribution can be obtained by putting q=0.25 in (6) as follows

$$X_{0.25}=\left[-2\theta^2 \ln\left[\frac{3}{3+a}\right]\right]^{\frac{1}{2\alpha}}.$$

Second quantile (Median) of the MOPR distribution is $X_{0.5}=\left[-2\theta^2 \ln\left[\frac{1}{1+a}\right]\right]^{\frac{1}{2\alpha}}$.

The third quantile of the MOPR distribution is $X_{0.75}=\left[-2\theta^2 \ln\left[\frac{1}{1+3a}\right]\right]^{\frac{1}{2\alpha}}$.

Some statistical measures can be found by quantile function as skewness S_K and kurtosis K_U measures. The S_K measure helps us to know to what degree and in which direction (positive or negative) the frequency distribution has a departure from symmetry, K_U gives a measure of flatness of distribution. These measures are less sensitive to outliers and they may exist for any distribution which does not have moments and can be defined based on quantile function, respectively as follows:

$$\mathrm{S_K}=\frac{Q(0.75)-2Q(0.5)+Q(0.25)}{Q(0.75)-Q(0.25)}$$

and $$\mathrm{K_U}=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(0.75)-Q(0.25)},$$

where $Q(.)$ is the quantile function. Table 2 introduces some outcomes for skewness and kurtosis for different values of parameters.

Table 2: Skewness and Kurtosis at $\theta=3$

	$\alpha=0.5$		$\alpha=0.8$		$\alpha=1$	
a	**S_K**	**K_U**	**S_K**	**K_U**	**S_K**	**K_U**
0.5	0.341	1.432	0.189	1.265	0.135	1.235
1	0.262	1.306	0.124	1.214	0.076	1.215
1.5	0.214	1.26	0.086	1.204	0.042	1.207
2	0.181	1.24	0.061	1.204	0.02	1.203
2.5	0.157	1.23	0.043	1.203	0.004	1.202

From Table 2,

i. As α and a increase, skewness and kurtosis decrease.

ii. the MOPR distribution is positive skewness and platykurtic

2.3 Linear representation

In this subsection we derived the representation of MOPR density function. Using Taylor's expansion and generalized binomial expansion, respectively as follows:

$$e^{-m\nu} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}(m\nu)^i, \quad (1-w)^{-s} = \sum_{j=0}^{\infty} \frac{\Gamma(s+j)}{\Gamma(s)\,\Gamma(j+1)} w^j ,$$

for $|w|<1, s>0$ (a real non-integer). The pdf (5) can be rewritten as:

$$f_{MOPR}(x,\underline{\xi}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i\, a\,\alpha}{2^i\,\theta^{2(i+1)} i!}(j+1)(1-a)^j\, x^{2\alpha(i+1)-1} e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right) j}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \eta_{i,j}\, \frac{a\,\alpha}{\theta^2}\, x^{2\alpha(i+1)-1} e^{-\left(\frac{j\,x^{2\alpha}}{2\theta^2}\right)}, \tag{7}$$

where

$$\eta_{i,j} = \frac{(-1)^i}{2^i\,\theta^{2i}\, i!}(j+1)(1-a)^j.$$

3. Statistical properties of the MOPR distribution

Some statistical properties of the MOPR distribution such as moments, moment generating function, incomplete moment and related measures are discussed in this section.

3.1 Non-Central moments

The r[th] non central moments $\mu_r^{'}$ of a continuous random variable X is defined by:

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

Using (8), we get the r[th] non central moments of the MOPR distribution as follows:

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \int_0^{\infty} \eta_{i,j}\, \frac{a\,\alpha}{\theta^2}\, x^{2\alpha(i+1)+r-1} e^{-\left(\frac{j\,x^{2\alpha}}{2\theta^2}\right)}\, dx$$

Taking $y = \left(\frac{j\,x^{2\alpha}}{2\theta^2}\right)$, then

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j} \left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+r}{2\alpha}\right)-1} \int_0^{\infty} y^{\left(\frac{2\alpha(i+1)+r}{2\alpha}\right)-2} e^{-y}\, dy$$

After solving the integral, we get

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\eta_{i,j} \left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+r}{2\alpha}\right)-1} \Gamma\left(\frac{2\alpha(i+1)+r}{2\alpha} - 1\right) \tag{8}$$

The first four non-central moments of the MOPR distribution can be calculated as follows:

For $r = 1, 2, 3, \text{and } 4$ respectively, in equation (8), we get:

$$\mu_1' = \mu = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}\right)-1}\Gamma\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right),$$

$$\mu_2' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+2}{2\alpha}\right)-1}\Gamma\left(\frac{2\alpha(i+1)+2}{2\alpha}-1\right),$$

$$\mu_3' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+3}{2\alpha}\right)-1}\Gamma\left(\frac{2\alpha(i+1)+3}{2\alpha}-1\right),$$

and

$$\mu_4' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+4}{2\alpha}\right)-1}\Gamma\left(\frac{2\alpha(i+1)+4}{2\alpha}-1\right).$$

3.2 Central moments

The k[th] central moments μ_k can be calculated through the following relation:

$$\mu_k = \int_{-\infty}^{\infty}(x-\mu)^k f(x)\,dx$$

Using (8), we get the k[th] central moments of the MOPR distribution as follows:

$$\mu_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^{\infty}\eta_{i,j}\frac{a\,\alpha}{\theta^2}(x-\mu)^k x^{2\alpha(i+1)-1}e^{-\left(\frac{j\,x^{2\alpha}}{2\theta^2}\right)}dx$$

Taking $y=\left(\frac{j\,x^{2\alpha}}{2\theta^2}\right)$, then

$$\mu_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{i}\int_0^{\infty}\left(\left(2\theta^2 j^{-1} y\right)^{1/2\alpha}-\mu\right)^k y^{i-1}e^{-y}\,dy$$

Using the series expansion, we have

$$\mu_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{k} a\,\mu^l\eta_{i,j}(-1)^l\left(\frac{2\theta^2}{j}\right)^{\left(\frac{k-l}{2\alpha}\right)+i}\binom{k}{l}\int_0^{\infty}y^{\left(\frac{k-l}{2\alpha}\right)+i-1}e^{-y}\,dy$$

After solving the integral, we get

$$\mu_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{k} a\,\mu^l\eta_{i,j}(-1)^l\left(\frac{2\theta^2}{j}\right)^{\left(\frac{k-l}{2\alpha}\right)+i}\binom{k}{l}\Gamma\left(\frac{k-l}{2\alpha}+i\right).$$

The first four central moments of the MOPR distribution can be calculated as follows:

$$\mu_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left[\left(\frac{2\theta^2}{j}\right)^{\frac{1}{2\alpha}+i}\Gamma\left(\frac{1}{2\alpha}+i\right)-\mu\left(\frac{2\theta^2}{j}\right)^{i}\Gamma(i)\right],$$

$$\mu_2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\,\eta_{i,j}\left[\left(\frac{2\theta^2}{j}\right)^{\frac{1}{\alpha}+i}\Gamma\left(\frac{1}{\alpha}+i\right)-2\,\mu\left(\frac{2\theta^2}{j}\right)^{\frac{1}{2\alpha}+i}\Gamma\left(\frac{1}{2\alpha}+i\right)+\mu^2\left(\frac{2\theta^2}{j}\right)^{i}\Gamma(i)\right]$$

$$=Variance(x)\ ,$$

$$\mu_3=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\,\eta_{i,j}\left[\begin{array}{l}\left(\frac{2\theta^2}{j}\right)^{\frac{3}{2\alpha}+i}\Gamma\left(\frac{3}{2\alpha}+i\right)-3\mu\left(\frac{2\theta^2}{j}\right)^{\frac{1}{\alpha}+i}\Gamma\left(\frac{1}{\alpha}+i\right)\\+3\mu^2\left(\frac{2\theta^2}{j}\right)^{\frac{1}{2\alpha}+i}\Gamma\left(\frac{1}{2\alpha}+i\right)-\mu^3\left(\frac{2\theta^2}{j}\right)^{i}\Gamma(i)\end{array}\right],$$

and

$$\mu_4=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\,\eta_{i,j}\left[\begin{array}{l}\left(\frac{2\theta^2}{j}\right)^{\frac{2}{\alpha}+i}\Gamma\left(\frac{2}{\alpha}+i\right)-4\mu\left(\frac{2\theta^2}{j}\right)^{\frac{3}{2\alpha}+i}\Gamma\left(\frac{3}{2\alpha}+i\right)\\+6\,\mu^2\left(\frac{2\theta^2}{j}\right)^{\frac{1}{\alpha}+i}\Gamma\left(\frac{1}{\alpha}+i\right)-4\mu^3\left(\frac{2\theta^2}{j}\right)^{\frac{1}{2\alpha}+i}\Gamma\left(\frac{1}{2\alpha}+i\right)\\+\mu^4\left(\frac{2\theta^2}{j}\right)^{i}\Gamma(i)\end{array}\right].$$

3.3 Moment generating function

The moment generating function $M_X(t)$ is generally defined as follows:

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}f(x)\,dx$$

Using (7), the moment generating function of the MOPR distribution is:

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r f(x,\underline{\xi})\,dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+r}{2\alpha}\right)-1}\left(\frac{t^r}{r!}\right)\Gamma\left(\frac{2\alpha(i+1)+r}{2\alpha}-1\right).$$

3.4 Incomplete moments

If a random variable is distributed according to MOPR distribution then its s[th] incomplete moments denoted by $I(t;s)$ can be calculated as follows:

$$I_{MOPR}(t;s)=\int_{-\infty}^{t}x^s f(x,\underline{\xi})\,dx\ =\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^t\eta_{i,j}\frac{a\,\alpha}{\theta^2}\,x^{2\alpha(i+1)+s-1}e^{-\left(\frac{j\,x^{2\alpha}}{2\theta^2}\right)}dx$$

After solving the integral, we get

$$I_{MOPR}(t;s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+s}{2\alpha}\right)-1}\Lambda\left(\frac{2\alpha(i+1)+s}{2\alpha}-1,\left(\frac{2\theta^2 t}{j}\right)^{\frac{1}{2\alpha}}\right), \tag{9}$$

where $\Lambda(t,s)=\int_0^t x^{s-1}e^{-x}\,dx$ is the lower incomplete gamma function.

3.5 Mean deviation

In statistics, the mount of variation in a population can be measured by the mean deviation of the mean and the median. δ_1 ,and δ_2 are represents the mean deviation about the mean μ and about the median M, respectively. According to random variable distributed MOPR, δ_1 and δ_2 are given by:

$$\delta_1=2\left[\mu F_{MOPR}(\mu)-I_{MOPR}(\mu;1)\right]$$

$$\delta_1=\left[\begin{array}{l}2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Gamma\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)\left[1-(1-a)\left(e^{-\left(\frac{\mu^{2\alpha}}{2\theta^2}\right)}\right)\right]^{-1}\\ \times\left(1-e^{-\left(\frac{\mu^{2\alpha}}{2\theta^2}\right)}\right)-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Lambda\left(\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right),\left(\frac{2\theta^2\mu}{j}\right)^{\frac{1}{2\alpha}}\right)\end{array}\right],$$

and

$$\delta_2=\mu-2I_{MOPR}(M;1),$$

$$\delta_2=\left[\begin{array}{l}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Gamma\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)\\ -2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\,\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Lambda\left(\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right),\left(\frac{2\theta^2 M}{j}\right)^{\frac{1}{2\alpha}}\right)\end{array}\right],$$

where $I_{MOPR}(.;.)$ is the first incomplete moment of MOPR distribution defined in (9). The income distributions are used some inequality measures called Lorenz and Bonferroni curves, which can be calculated using incomplete moment. For MOPR distribution, the Lorenz $L(p)$ and Bonferroni $B(p)$curves are, respectively, given as follows:

$$L_{MOPR}(p)=\frac{I_{MOPR}(p;1)}{\mu}$$

$$=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Lambda\left(\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right),\left(\frac{2\theta^2 p}{j}\right)^{\frac{1}{2\alpha}}\right)}{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a\eta_{i,j}\left(\frac{2\theta^2}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Gamma\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)},$$

and

$$B_{MOPR}(P)=\frac{L_{MOPR}(P)}{F_{MOPR}(x)}$$

$$=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^{2}}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Lambda\left(\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right),\left(\frac{2\theta^{2}p}{j}\right)^{\frac{1}{2\alpha}}\right)}{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a\,\eta_{i,j}\left(\frac{2\theta^{2}}{j}\right)^{\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)}\Gamma\left(\frac{2\alpha(i+1)+1}{2\alpha}-1\right)\left(1-e^{-\left(\frac{x^{2\alpha}}{2\theta^{2}}\right)}\right)\left[1-(1-a)\left(e^{-\left(\frac{x^{2\alpha}}{2\theta^{2}}\right)}\right)\right]^{-1}}.$$

3.6 Rényi entropy measure

The Rényi entropy is used to quantify the uncertainty of variation in a random variable X. The Rényi entropy measure of a continuous random variable X of order δ is given by:

3.6 $$I_R(\delta)=\frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty} f^{\delta}(x)\,dx\right],\quad \delta\geq 0,\ \delta\neq 1.$$

Hence, Rényi entropy measure of a random variable X distributed MOPR will be given by:

$$I_R(\delta)=\frac{1}{1-\delta}\log\left[\int_{0}^{\infty} f_{MOPR}^{\delta}(x)\,dx\right],$$

Using (7), the Rényi entropy measure of the MOPR distribution is:

$$I_R(\delta)=\frac{1}{1-\delta}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\eta_{i,j}\frac{1}{2\alpha}\left(\frac{a\alpha}{\theta^{2}}\right)^{\delta}\left(\frac{2\theta^{2}}{\delta j}\right)^{\left((i+1)-\frac{1}{2\alpha}\right)\delta+\frac{1}{2\alpha}}\Gamma\left(\frac{1}{2\alpha}+\left((i+1)-\frac{1}{2\alpha}\right)\delta\right)\right]$$

3.7 Order statistics

Order statistics arise frequently in the reliability, and life testing. In this subsection, for MOPR distribution we calculate the pdf of the k^th order statistics, smallest order statistics, and largest order statistics. Also, the moment of the k^th order statistics are derived. Suppose $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ be an order statistics sample corresponding to n-sized random sample from the MOPR distribution with cdf (4) and pdf (5). The pdf of $X_{(k)}$ can be expressed as:

$$f_{X_{(k)}}(x)=\sum_{t=0}^{n-k}\frac{(-1)^{t}}{B(k,n-k+1)}\binom{n-k}{t}\left[F_{MOPR}(x)\right]^{k+t-1} f_{MOPR}(x). \qquad (10)$$

The $(k+t-1)^{th}$ power of the cdf in (10) in series representation is given by:

$$\left[F_{MOPR}(x)\right]^{k+t-1}=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{m+l}\binom{i+j-1}{m}\binom{1-i-j}{l}(1-a)^{l}e^{-\left(\frac{(m+l)x^{2\alpha}}{2\theta^{2}}\right)}. \qquad (11)$$

Substituting (7) and (11) into (10), we get:

$$f_{X_{(k)}}(x)=\frac{a\,\alpha}{\theta^2 B(k,n-k+1)}\,\psi_{t,i,j,m,l}\,e^{-(m+l+j)\frac{x^{2\alpha}}{2\theta^2}}, \tag{12}$$

Equations (13) and (14) are represents the pdf of the smallest and largest MOPR order statistics respectively,

$$f_{X_{(1)}}(x)=\frac{n\,a\,\alpha}{\theta^2}\,\psi^*_{t,i,j,m,l}\,e^{-(m+l+j)\frac{x^{2\alpha}}{2\theta^2}}, \tag{13}$$

and

$$f_{X_{(n)}}(x)=\frac{n\,a\,\alpha}{\theta^2}\,\psi_{i,j,m,l}\,e^{-(m+l+j)\frac{x^{2\alpha}}{2\theta^2}}, \tag{14}$$

Furthermore, from (12), The moments of the kth order statistics MOPR distribution is given by:

$$E(X_{(k)}^{p})=\frac{a}{(m+l+t)}\left(\frac{2\theta^2}{(m+l+t)}\right)^{i+\frac{p}{2\alpha}}\psi_{t,i,j,m,l,n}\,\Gamma\left(i+\frac{p}{2\alpha}+1\right),$$

where

$$\psi_{t,i,j,m,l}=\sum_{t=0}^{n-k}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{m+l+t}\eta_{i,j}\binom{n-k}{t}\binom{i+j-1}{m}\binom{1-i-j}{l}(1-a)^l\,x^{2\alpha(i+1)-1},$$

$$\psi^*_{t,i,j,m,l}=\sum_{t=0}^{n-1}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{m+l+t}\eta_{i,j}\binom{n-1}{t}\binom{i+j-1}{m}\binom{1-i-j}{l}(1-a)^l\,x^{2\alpha(i+1)-1},$$

$$\psi_{i,j,m,l}=\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{m+l}\eta_{i,j}\binom{i+j-1}{m}\binom{1-i-j}{l}(1-a)^l\,x^{2\alpha(i+1)-1}.$$

$$\psi_{t,i,j,m,l,n}=\sum_{t=0}^{n-k}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{m+l+t}}{B(k,n-k+1)}\eta_{i,j}\binom{n-k}{t}\binom{i+j-1}{m}\binom{1-i-j}{l}(1-a)^l\,x^{2\alpha(i+1)-1}.$$

4. Characterization results

A characterization of probability distributions played an important role in distribution theory and statistical studies of sciences and applied sciences. Here, we provide certain characterizations of MOPR distribution based on a relation between two truncated moments and in terms of the reversed hazard function. It is also considered one of the benefits of characterization that these results are also achieved in the following cases, when the interval of distribution is not closed and when the cdf does not have closed form, see Glanzel (1990). Here, we provide an important theorem due to Glanzel (1987) as a tool for the characterization of the MOPR distribution.

Theorem 4.1

Suppose that (Ω,F,P) be a given probability space and let H = [a, b] be an interval for some a< b . Let X : Ω → H be a continuous random variable. with cdf F. Also, suppose h and g are two real functions defined on H, where

$$E[g(X)|X \geq x] = E[h(X)|X \geq x].\eta(x), \quad x \in H$$

defined with some real function η. Assume that $h, g \in C^1(H), \eta \in C^2(H)$, and F is twice continuously differentiable and strictly monotone function on the set H.

Finally, assume that the equation $\eta(x)h(x) = g(x)$, has no real solution in the interior of H. Then F is uniquely determined by the functions h, g and η, specifically

$$F(x) = \int_0^x K \left| \frac{\eta'(y)}{\eta(y)h(y) - g(y)} \right| e^{-s(y)} dy,$$

where the function s is the solution of the differential equation $s'(x) = \dfrac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)}$, and K is the normalized constant, such that $\int_H dF = 1$. We provide our characterizations in following subsections.

4.1 Characterizations based on truncated moments

Characterizations of MOPR distribution in terms of a simple relationship between two truncated moments are presented in this subsection.

Proposition 4.1

Suppose that X: $\Omega \to [0, \infty[$ be a continuous random variable. Also, let h and g are two real functions defined as follows:

$$h(x) = \left[a + (1-a)\left(1 - e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)\right]^2, \text{ and } g(x) = h(x)e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}, \; x > 0.$$

The random variable X has pdf of MOPR if and only if the function η has the form $\eta(x) = \frac{1}{2}e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}$, where η is defined in Theorem 4.1.

Proof

Let X be a MOPR random variable with pdf (5), then

$$(1 - F(x))E[h(X)|X \geq x] = a\,e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}, \; x > 0,$$

$$(1 - F(x))E[g(X)|X \geq x] = \frac{a}{2}e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}, \; x > 0,$$

and

$$\eta(x)h(x) - g(x) = \frac{-1}{2}h(x)e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)} < 0, \text{ for } x > 0.$$

On the other hand, if η is given as above, then

$$s'(x) = \frac{\eta'(x)h(x)}{\eta(x)h(x) - g(x)} = \left(\frac{\alpha}{\theta^2}\right)x^{2\alpha-1}, \; x > 0.$$

According to Theorem 4.1, X has pdf (5).

Corollary 4.1

Let X: $\Omega \rightarrow [0,\infty[$ be a continuous random variable and let the function h be as in Proposition 4.1. The pdf of X is (5) if and only if there exist functions g and η defined in Theorem 5.1 satisfying the differential equation:

$$\frac{\eta'(x)\,h(x)}{\eta(x)\,h(x)-g(x)}=\left(\frac{\alpha}{\theta^2}\right)x^{2\alpha-1} \ ,\ x>0,$$

with general solution as follows:

$$\eta(x)=e^{\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\left[-\int\left(\frac{\alpha}{\theta^2}\right)x^{2\alpha-1}\,e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\,h^{-1}(x)\,g(x)\,+C\right],$$

where C is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with C = 0. Noted that there are other triplets (h,g,η) verifying the conditions of Theorem 4.1.

4.2 Characterizations based on reversed hazard function

In this subsection we present a characterization of the MOPR distribution, for $a=1$, in terms of the reverse hazard function. The reversed hazard function, $r_F(x)$, of a twice differentiable distribution function, F, is defined as

$$r_F(x)=\frac{f(x)}{F(x)}.$$

Proposition 4.2

Let X: $\Omega \rightarrow [0,\infty[$ be a continuous random variable. The random variable X has pdf (5) for $a=1$, if and only if its reverse hazard function $r_F(x)$ verifies the following differential equation:

$$r_F'(x)+\left(\frac{\alpha}{\theta^2}\right)\,x^{2\alpha-1}\,r_F(x)=\left(\frac{\alpha}{\theta^2}\right)e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\frac{d}{dx}\left(\frac{x^{2\alpha-1}}{1-e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}}\right),\ \lim_{x\to\infty} r_F(x)=0.$$

Proof

Let X be a MOPR random variable with pdf (5), then clearly the above differential equation holds. On the other hand, if the differential equation holds, then

$$\frac{d}{dx}\left[e^{\left(\frac{x^{2\alpha}}{2\theta^2}\right)}r_F(x)\right]=\left(\frac{\alpha}{\theta^2}\right)\frac{d}{dx}\left(\frac{x^{2\alpha-1}}{1-e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}}\right),\ x>0.$$

Then

$$r_F(x)=\left(\frac{\alpha}{\theta^2}\right)\,x^{2\alpha-1}\,e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\left(1-e^{-\left(\frac{x^{2\alpha}}{2\theta^2}\right)}\right)^{-1}.$$

which is the reverse hazard function corresponding to the pdf (5) for $a=1$.

5. Statistical inference

In this section, we shall discuss the method of maximum likelihood estimation for estimating the parameters of the MOPR distribution. The method's success stems no doubt from its many asymptotic properties for estimated parameters which are often utilized to obtain confidence intervals (CI) and test of statistical hypotheses. Suppose that $\underline{x} = (x_1, x_2, \ldots, x_n)$ is a random sample from the MOPR distribution with pdf (5). Then the log-likelihood function, denoted by $\ln l$, for of the MOPR distribution with parameter $\underline{\xi} = (a, \alpha, \theta)$ can be written as:

$$\ln l = n \ln\left(\frac{a\alpha}{\theta^2}\right) + \sum_{i=1}^{n} \ln\left(x_i^{2\alpha-1}\right) - \sum_{i=1}^{n}\left(\frac{x_i^{2\alpha}}{2\theta^2}\right) + \sum_{i=1}^{n} \ln\left[1-(1-a)\,e^{-\left(\frac{x_i^{2\alpha}}{2\theta^2}\right)}\right]^{-2}, \qquad (15)$$

Differentiate (15) with respect to unknown parameters and then equate by zero, we get

$$\frac{n}{\hat{a}} - 2\sum_{i=1}^{n} e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)} \left[1-(1-\hat{a})\,e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right]^{-1} = 0,$$

$$\frac{-2n}{\hat{\theta}} + \sum_{i=1}^{n}\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\right) + 2\sum_{i=1}^{n}(1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\right) e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)} \left[1-(1-\hat{a})\,e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right]^{-1} = 0,$$

and

$$\frac{n}{\hat{\alpha}} + 2\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n}\left(\frac{x_i^{2\hat{\alpha}} \ln x_i}{\hat{\theta}^2}\right) - 2\sum_{i=1}^{n}(1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}} \ln x_i}{\hat{\theta}^2}\right) e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)} \left[1-(1-\hat{a})\,e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right]^{-1} = 0,$$

Solve the desired system of equations by some numerical methods, to get maximum likelihood estimates (MLEs) $\hat{\underline{\xi}} = (\hat{a}, \hat{\theta}, \hat{\alpha})$. The interval estimates of the parameters are obtained by first finding the approximate information matrix, whose elements are negative of the expected values of the second order derivative of logarithms of the likelihood function but it may be obtained by replacing expected values by their MLEs (see Cohen (1965)). Therefore, the elements of the approximate information matrix are given by:

$$\frac{-\partial^2 \ln l}{\partial a^2} = \frac{n}{\hat{a}^2} - 2\sum_{i=1}^{n}\left(e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right)^2 \hat{w}_i^{-2},$$

$$\frac{-\partial^2 \ln l}{\partial a\,\partial\theta} = 2\sum_{i=1}^{n}\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\right) e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)} \hat{w}_i^{-2},$$

$$\frac{-\partial^2 \ln l}{\partial a\,\partial\alpha} = -2\sum_{i=1}^{n}\left(\frac{x_i^{2\hat{\alpha}} \ln x_i}{\hat{\theta}^3}\right) e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)} \hat{w}_i^{-2},$$

$$\frac{-\partial^2 \ln l}{\partial \theta^2} = \frac{-2n}{\hat{\theta}} + 3\sum_{i=1}^{n}\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^4}\right) - 2\sum_{i=1}^{n}(1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\right)e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\left[\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\right)-\frac{3}{\hat{\theta}}\right]\hat{w}_i^{-1}$$

$$-2\sum_{i=1}^{n}\left((1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\right)e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right)^2\hat{w}_i^{-2},$$

$$\frac{-\partial^2 \ln l}{\partial \theta\, \partial \alpha} = -2\sum_{i=1}^{n}\left(\frac{x_i^{2\hat{\alpha}}\ln x_i}{\hat{\theta}^3}\right) - 4\sum_{i=1}^{n}(1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^3}\ln x_i\right)e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\left[1-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)\right]\hat{w}_i^{-1}$$

$$+\frac{2}{\hat{\theta}}\sum_{i=1}^{n}\ln x_i\left((1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^2}\right)e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right)^2\hat{w}_i^{-2},$$

and

$$\frac{-\partial^2 \ln l}{\partial \alpha^2} = \frac{n}{\hat{\alpha}^2} + 2\sum_{i=1}^{n}\left(\frac{x_i^{\hat{\alpha}}\ln x_i}{\hat{\theta}}\right)^2 + 2\sum_{i=1}^{n}(1-\hat{a})\left(\frac{x_i^{\hat{\alpha}}\ln x_i}{\hat{\theta}}\right)^2 e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\left[2-\left(\frac{x_i^{2\hat{\alpha}}}{\hat{\theta}^2}\right)\right]\hat{w}_i^{-1}$$

$$+2\sum_{i=1}^{n}\left((1-\hat{a})\left(\frac{x_i^{2\hat{\alpha}}\ln x_i}{\hat{\theta}^2}\right)e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right)^2\hat{w}_i^{-2},$$

where $\hat{w}_i = \left[1-(1-\hat{a})\,e^{-\left(\frac{x_i^{2\hat{\alpha}}}{2\hat{\theta}^2}\right)}\right].$

Under certain regularity conditions, the asymptotic variance-covariance matrix for the MLEs can be obtained by inverting the information matrix. The approximate $100(1-\eta)\%$ two-sided confidence intervals for $\underline{\xi} = (a,\theta,\alpha)$ can be respectively constructed as,

$$\hat{a} + Z_{\eta/2}\sqrt{Var(\hat{a})},\ \hat{\theta} + Z_{\eta/2}\sqrt{Var(\hat{\theta})},\ \text{and}\ \hat{\alpha} + Z_{\eta/2}\sqrt{Var(\hat{\alpha})},$$

where $Z_{\eta/2}$ is the upper percentile of standard normal distribution.

6. Simulation illustration

In this section, we shall investigate the accuracy of the MLEs of parameters of the MOPR distribution based on certain measures, which are biase (B), mean square errors (MSEs) and variance for different sample sizes. The inversion method is used to generate samples, i.e., the random samples of sizes n=50(50)250 having the MOPR distribution are generated using (6). The number of replications is 1000 times and considers different values of parameters (a,α,θ) are chosen. All computations of the simulation study were performed using Mathcad package software. The simulation results of calculating the B, MSE, and variance values are given in Table 3. From Table (3), the values of MSE and variances decreases as the sample size n increases. Thus, it can be concluded that the MLEs of MOPR are consistent.

7. Real data illustration

The applicability of MOPR distribution is illustrated using engineering science real-world data sets in this section. Comparison of the proposed MOPR distribution has been made with the Rayleigh (R) distribution, and Power Rayleigh (PR) distribution with the help of two real data sets.

To test the superiority of the MOPR distribution, some goodness of fit statistics measures can be used in comparison to some other distributions using the computational package Mathcad, namely $-2\ln l$, Kolmogorov-Smirnov (KS) and its p-value, Anderson-Darling (A^{*}), Cramer-von Mises (W^{*}), and Liao-Shimokawa (L-S) statistics. These statistics can determine how closely the MOPR distribution fit the empirical distribution of the data. The distribution with better fit than the others will be the one having the largest *p*-value and the smallest values for statistics.

Table 3: Simulated results of the MOPR distribution.

n	$(a=0.3,\ \alpha=0.2,\ \theta=0.3)$			$(a=0.3,\ \alpha=0.2,\ \theta=0.7)$		
	B	MSE	Variance	B	MSE	Variance
50	0.111 1.982 7.828E(-4)	0.013 2.375 4.393E(-5)	0.012 0.847 4.501E(-5)	0.438 2.249 0.021	0.214 2.129 4.995E(-4)	2.729E(-4) 0.572 4.981E(-5)
100	0.110 0.976 3.369E(-4)	0.012 2.139 3.751E(-5)	9.799E(-3) 0.447 3.689E(-5)	0.363 1.154 0.012	0.114 1.788 1.962E(-4)	1.598E(-4) 0.457 1.478E(-5)
150	0.108 0.349 3.287E(-4)	0.011 0.327 2.454E(-5)	3.574E(-3) 0.205 2.341E(-5)	0.325 0.476 6.967E(-3)	0.096 0.266 5.514E(-5)	2.786E(-5) 0.039 6.595E(-6)
200	0.106 0.244 1.826E(-4)	0.010 0.053 1.618E(-5)	2.209E(-3) 0.112 1.585E(-5)	0.316 0.321 5.107E(-3)	0.094 0.163 2.982E(-5)	1.698E(-5) 0.011 3.734E(-6)
250	0.105 0.139 1.131E(-4)	4.157E(-3) 0.043 1.031E(-5)	1.259E(-3) 0.024 1.097E(-5)	0.301 0.315 2.461(-3)	0.091 0.111 8.194E(-6)	1.132E(-5) 0.065 2.137E(-6)
n	$(a=0.1,\ \alpha=0.2,\ \theta=0.5)$			$(a=0.3,\ \alpha=0.2,\ \theta=0.5)$		
	B	MSE	Variance	B	MSE	Variance
50	0.095 2.101 7.748E(-4)	8.874E(-3) 2.221 2.096E(-3)	9.069E(-3) 1.978 3.685E(-5)	0.142 2.249 0.196	0.028 1.057 5.211E(-4)	6.415E(-4) 1.921 4.859E(-5)
100	0.094 1.117 3.374E(-4)	8.497E(-3) 1.392 3.741E(-5)	3.573E(-4) 0.998 1.027E(-5)	0.114 1.234 0.193	0.018 0.977 2.149E(-4)	1.277E(-4) 0.533 1.519E(-5)
150	0.092 0.509 3.821E(-4)	8.327E(-3) 0.254 1.029E(-5)	9.797E(-5) 0.409 4.395E(-6)	0.113 0.536 0.192	0.014 0.324 7.871E(-5)	2.666E(-5) 0.037 7.177E(-6)
200	0.090 0.446 1.851E(-4)	8.199E(-3) 0.134 2.454E(-6)	4.158E(-5) 0.022 2.341E(-6)	0.111 0.401 0.186	0.012 0.224 5.023E(-5)	1.657E(-5) 0.077 4.256E(-6)
250	0.089 0.336 1.185E(-4)	1.258E(-5) 0.150 1.616E(-6)	2.209E(-5) 0.022 1.582E(-6)	0.103 0.383 0.178	0.011 0.178 1.512E(-5)	1.126E(-5) 0.017 2.417E(-6)

Carbon fibres real data sets

1. The first data consists of the breaking stress of 66 carbon fibres of 50 mm length (GPa) and used by Aqtash et al. (2014). The data set:

0.39	1.61	1.89	2.41	2.55	2.79	2.93	3.11	3.27	3.39	3.75
0.85	1.61	2.03	2.43	2.56	2.81	2.95	3.15	3.28	3.56	4.20
1.08	1.69	2.03	2.48	2.59	2.82	2.96	3.15	3.31	3.60	4.38
1.25	1.80	2.05	2.50	2.67	2.85	2.97	3.19	3.31	3.65	4.42
1.47	1.84	2.12	2.53	2.73	2.87	3.09	3.22	3.33	3.68	4.70
1.57	1.87	2.35	2.55	2.74	2.88	3.11	3.22	3.39	3.70	4.90

Table 4 provides the MLEs of parameters of the R, PR and MOPR distributions. The statistics measures for distributions are mentioned in Table 5.

Table 4: The MLEs of parameters for the first data.

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{a}$
R (θ)	2.049	-	-
PR (θ,α)	4.850	1.721	-
MOPR (a,θ,α)	0.901	0.811	5.379

Table 5: Statistics Measures for the first data.

Distribution	$-2\ln l$	**A***	**W***	**L-S**	**K-S**	**P-Value**
R (θ)	196.42	16.73	15.74	5.07	0.23	0.002
PR (θ,α)	172.14	10.52	13.57	0.594	0.08	0.763
MOPR (a,θ,α)	**169.431**	**0.256**	**0.037**	**0.225**	**0.009**	**1**

Based Table 5, The MOPR distribution has the smallest values of the statistical measures than other fitted distributions, so it provides best fit for carbon fibres data.

2. The second data set consists of the strength data measured in (GPa) of 69 single carbon fibres tested under tension at gauge lengths of 20 mm, which reported by Badar and Priest (1982). For illustrative purpose, we are considering the same transformed data set as taken by Raqab and Kundu (2005). The transformed data set:

0.312	0.861	1.006	1.140	1.272	1.426	1.514	1.629	1.726	1.821
0.314	0.865	1.021	1.179	1.274	1.434	1.535	1.633	1.770	1.848
0.479	0.944	1.027	1.224	1.301	1.435	1.554	1.642	1.773	1.880
0.552	0.958	1.055	1.240	1.359	1.478	1.566	1.648	1.800	1.954
0.700	0.966	1.063	1.253	1.382	1.490	1.570	1.684	1.809	2.012
0.803	0.977	1.098	1.270	1.382	1.511	1.586	1.697	1.818	2.067
2.084	2.090	2.096	2.128	2.233	2.433	2.585	2.585		

Table 6 presentes the MLEs of parameters of the MOPR and competitive distributions. Table 7 introduced the statistics measures for distributions.

Table 6: The MLEs of parameters for the second data.

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{a}$
R (θ)	1.083	-	-
PR (θ,α)	1.543	1.623	-
MOPR (a,θ,α)	0.754	1.052	6.155

Table 7: Statistics Measures for the second data.

Distribution	$-2\ln l$	A*	W*	L-S	K-S	P-Value
R (θ)	118.84	34.49	6.342	0.428	0.19	0.008
PR (θ,α)	98.06	37.148	6.567	0.48	0.04	0.999
MOPR (a,θ,α)	**97.393**	**0.153**	**0.016**	**0.326**	**0.013**	**1**

Based on Table 7, MOPR distribution fits the second carbon fibres data set best. In general, From the results of the two datasets proposed MOPR distribution performs the best distribution according to the other competitive distributions.

8. Conclusion

A new three-parameter of the Power Rayleigh distribution based on Marshall olkin generated family (1997) called MOPR distribution has been studied in detail. The proposed distribution has several sub-models as special cases. Some mathematical properties of the proposed distribution accompanied with characterizations based on truncated moments and reverse hazard function are discussed. The MOPR parameters are estimated by maximum likelihood method. Simulation results are provided to assess the performance of the proposed estimated parameters. Real data sets illustrate the flexibility and usefulness of the proposed model with other competitive models. The results confirms that the MOPR distribution has better fitting carbon fibers data in comparison of sub-models such as PR, and R distribution

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