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Beta restricted regression estimators: Simulation and application

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Beta regression model; Lagrange Multiply; Monte-Carlo simulation; Prior Information; Restricted regression.

Abstract

Statistics is the science that aims to collect and analyze data. In data analysis, researchers need to collect all information that serves their study. Using full information about the parameter leads to fitting an appropriate model for the data that researchers collect. This study aimed to fit a constrained beta regression model using prior information (BCML). Mean square error (MSE) has been used to justify the new estimator. Real data and simulation have been done using R.4.2.2. Results indicate that the constrained beta regression is better than the standard beta regression, where its MSE was less.

1. Introduction

Prior Information analyzes the data better. Ignoring this prior information may affect the decisions made by researchers. This study used the beta regression model because of its importance and widely spread. Analyzing data that come in the form of rates or percentages needs distribution that its ranges lie between zero and one like a beta regression model (Abonazel and Algamal, 2021), (Abonazel et al, 2022a), and (Abonazel et al, 2022b). If there exists additional information from these models about the estimator beta, restricted regression estimators must be used to get a good fit. Previous studies like (Abonazel and Algamal, 2021) discuss ridge beta regression to handle multicollinearity problems but don't have constrained beta regression. This paper proposes a constrained estimator for the regression of the beta model to get more fit of the beta regression model. The traditional Maximum likelihood (ML) method has been used to estimate parameters with and without the prior information (Constrained estimator).

ML estimator has been used to check the performance of constrained estimators MSE criteria has been used to compare between the traditional beta regression estimator and Beta restricted regression estimators. A simulation study and an empirical application have been done to make a good decision about the new estimator. Simulation and application results indicate that the Beta restricted regression estimator BCML is better than the ML estimator in terms of MSE where its MSE was less.

Section 2 illustrates the beta regression model and its estimation using the ML method. It also indicates the BCML estimator. Numerical evaluation has been proposed in Sections 3 and 4 using both simulation and real data application. Finally, conclusions have been presented in Section 5.

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2. Methodology

ML is introduced in this section. Then the maximum likelihood for beta regression is introduced. The constrained linear model has been explained in the next part. Finally, the constrained estimator for the beta regression model is discussed with its properties.

2.1 Model of Beta regression

Researchers can use the model of beta regression in case their study contains dependent variables represented as percentages or ratios. In the case of the dependent variable was continuous and ranged from zero to one, the beta regression model exists to estimate parameters and get a good estimator for data. It has become popular in many fields as medical studies, for example, the model of beta regression can be used to measure the percentage of Diabetes in the blood in medical studies. Beta regression was used to study the effect of some independent variables in the case of the dependent variable ranging from zero to one. (Ferrari and Cribari-Neto, 2004) introduced the beta regression model, a link function used in the model to link between the dependent and independent variables. Dispersion measure has been included in the model as a precision parameter. The standard form of the model of beta regression has been introduced in this paper; where the precision parameter is assumed to be constant across observations. However, it may not be constant across observations (Abonazel and Taha, 2021).

If the random variable y is continuous and ranges between zero and one, y can be considered to follow the beta distribution. The probability density function of beta (pdf) can be written as:

$$f(y,\mu,\varphi)=\frac{\Gamma(\varphi)}{\Gamma(\mu\varphi)\Gamma((1-\mu)\varphi)}y^{\mu\varphi-1}(1-y)^{\varphi-1(1-\mu)};\ 0<\mu,y<1 \text{ and } \varphi>0 \tag{1}$$

where $\Gamma.$ is the gamma function, (Bayer and Cribari-Neto, 2017) compute precision parameter φ as:

$$\varphi=\frac{1-\sigma^2}{\sigma^2}$$

The mean and the variance of the random variable y that follows the beta distribution can be written as:

$$E(y)=\mu, var(y)=\ \mu(1-\mu)\sigma^2$$

where the model of μ_i can be represented as:

$$g(\mu_i)=log\left(\frac{\mu_i}{1-\mu_i}\right)=X^T\beta=\eta \tag{2}$$

where $\beta=(\beta_1,\beta_2,\dots,\beta_p)^{/}$ is a $(p \text{ x } 1)$ vector of unknown parameters, η is the linear predictor, and X is an $(n \text{ x } p)$ matrix of repressors. $g(.)$ is a monotonic differentiable link function used to relate the systematic component with the random component.

2.2 The estimator of Maximum likelihood

The function of the log-likelihood for the model of beta regression can be written as (Qasim and et.al, 2021):

$$\boldsymbol{\ell}(\boldsymbol{\beta}) = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{\mathcal{L}}(\mu_i, \varphi_i; \boldsymbol{y_i}) = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \{\boldsymbol{log}\Gamma(\varphi_i) - \boldsymbol{log}\Gamma(\mu_i\varphi_i) - \boldsymbol{log}\Gamma((1-\mu_i)\varphi_i) + (\mu_i\varphi_i - 1)log(y_i) + ((1-\mu_i)\varphi_i - 1)log(1-y_i)\} \quad (3)$$

the score function for β can be obtained by differentiating the log-likelihood function in Eq. (3) as:

$$U(\beta) = \varphi X^{/} T(y^* - \mu^*), \quad (4)$$

Such that:

$$T = diag\left(\frac{1}{g^{/}(\mu_1)}, \dots, \frac{1}{g^{/}(\mu_n)}\right), y^* = (y_1^*, \dots, y_n^*)^{/}, \mu^* = (\mu_1^*, \dots, \mu_n^*)^{/}, y_i^* = log\left(\frac{y_i}{1-y_i}\right), and\ \mu_i^* = \Psi\big(\mu_i\varphi_i - \Psi((1-\mu_i)\,\varphi_i)\big)$$

where: $\Psi(.)$ denoting the digamma function. The Fisher scoring algorithm used for estimating β estimated by (Espinheira, da Silva, and Silva 2015; Espinheira et al. 2019) as:

$$\beta^{(m+1)} = \beta^{(m)} + \left(I_{(\beta\beta)}^{(m)}\right)^{-1} U_\beta^{(m)}(\beta),$$

where the score function defined in Eq. (4) is $U_\beta^{(m)}$ (Espinheira et al, 2019), and $I_{(\beta\beta)}^{(m)}$ is the information matrix for β. The least squares method can be used to get the initial value of β, while the initial value for each precision parameter can be calculated as:

$$\widehat{\varphi_i} = \frac{\hat{\mu}_i(1-\hat{\mu}_i)}{\hat{\sigma}_i^2} \quad (5)$$

where $\hat{\mu}_i$ and $\hat{\sigma}_i^2$ values are obtained from linear regression, the ML estimator of β is obtained as:

$$\hat{\beta}_{ML} = \left(X^T \widehat{W} X\right)^{-1}\left(X^T \widehat{W} Z\right) \quad (6)$$

Such that $Z = \hat{\eta} + \widehat{W}^{-1}\hat{T}(y^* - \mu^*)$, and $\widehat{W} = diag(\widehat{W}_1, \dots, \widehat{W}_n)$;

$$\widehat{W}_i = \varphi_i\left\{\Psi^{/}\left(\mu_i\varphi_i + \Psi^{/}((1-\mu_i)\,\varphi_i)\right)\right\}\frac{1}{\{g^{/}(\hat{\mu}_i)\}^2}$$

since, $\widehat{W}$ and $\hat{T}$ are the matrices W and T, respectively, evaluated at the ML estimator. The ML estimator of β is normally distributed with asymptotic mean vectors $E(\hat{\beta}_{ML}) = \beta$, and asymptotic covariance matrix:

$$cov(\hat{\beta}_{ML}) = \frac{1}{\varphi}\left(X^{/}\widehat{W}X\right)^{-1} \quad (7)$$

the asymptotic trace of mean squared error (MSE) for the ML estimator can be calculated as:

$$MSE(\hat{\beta}_{ML}) = tr\left[\frac{1}{\varphi}\left(X^{/}\widehat{W}X\right)^{-1}\right] = \frac{1}{\varphi}\sum_{j=1}^{p}\frac{1}{\lambda_j} \quad (8)$$

2.3 Linear and ordinary mixed restricted estimator

The linear regression model takes the form:

$$\boldsymbol{Y_{n.1}} = \boldsymbol{X_{n.p}\beta_{p.1}} + \epsilon_{\boldsymbol{n.1}} \quad (9)$$

, the ordinary least squares (OLS) estimator of β can be written as:

$$\boldsymbol{\hat{\beta}} = (\boldsymbol{X^T X})^{-1}\boldsymbol{X^T Y} \tag{10}$$

It is distributed by normal $\mathcal{N}(\boldsymbol{\beta}, \boldsymbol{\sigma^2}(\boldsymbol{X^T X})^{-1})$, where the column vectors in $\boldsymbol{X}$ are linearly independent. The restricted model for $\boldsymbol{\hat{\beta}}$ can be written as $\boldsymbol{r} = \boldsymbol{R\beta}$ where $\boldsymbol{R}$ is an $\boldsymbol{q}$ x $\boldsymbol{p}$ matrix($\boldsymbol{q} \leq \boldsymbol{p}$), and $\boldsymbol{r}$ is $\boldsymbol{q}$ x $\boldsymbol{1}$ vector of restrictions, the restricted parameter $\boldsymbol{\beta^C}$ using the Lagrange function is given by:

$$L = (Y - X\beta)^T(Y - X\beta) + \lambda(r - R\boldsymbol{\hat{\beta}})$$

$$\frac{\partial}{\partial\boldsymbol{\hat{\beta}}}L = -2X^TY + 2(X^TX)\,\beta^C + R^T\lambda = 0$$

$$\frac{\partial}{\partial\lambda}L = r - R\beta^C = 0$$

$$\lambda = -2(R(X^TX)^{-1}R^T)^{-1}(r - R\hat{\beta})$$

$$\boldsymbol{\beta^C} = \boldsymbol{\hat{\beta}} + (\boldsymbol{X^T X})^{-1}\boldsymbol{R^T}(R(\boldsymbol{X^T X})^{-1}R^T)^{-1}(\boldsymbol{r} - \boldsymbol{R\hat{\beta}}) \tag{11}$$

The combination of the LM and the restricted model introduces the Ordinary Mixed Estimator (OME) proposed by (Theil and Goldberger, 1961) as follows:

$$\begin{pmatrix} Y \\ r \end{pmatrix} = \begin{pmatrix} X \\ R \end{pmatrix}\beta + \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} \tag{12}$$

where: $\mathbb{E}\left\{\begin{pmatrix} \epsilon \\ \epsilon^* \end{pmatrix}(\epsilon^/ \quad \epsilon^{*/})\right\} = \sigma^2\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$

The OME was unbiased where $\mathbb{E}(\hat{\beta}_{OME}) = \beta$, the variance was given by $var(\hat{\beta}_{OME}) = \sigma^2(X^/X + R^/R)^{-1}$ (Abdemegaly, 2019), and the mean square error $\text{MSE}(\hat{\beta}_{OME}) = \sigma^2(X^/X + R^/R)^{-1}$.

2.4 Beta restricted estimator

In the case of the dependent variable taking percentages, linear model and normal distribution are considered to be non-useful to fit the model. According to previous works, no studies have been introduced to discuss the constrained model of beta regression. The constrained beta (BCML) estimator can be written as using the previous studies:

$$\boldsymbol{\beta^C_{ML}} = \boldsymbol{\hat{\beta}_{ML}} + (\boldsymbol{x^{*/}W\,x^*})^{-1}\boldsymbol{R^{*/}}(\boldsymbol{R}(\boldsymbol{x^{*/}\,Wx^*})^{-1}\boldsymbol{R^/})^{-1}(\boldsymbol{r} - \boldsymbol{R\hat{\beta}_{ML}}) \tag{13}$$

To illustrate the main basic steps of the $\boldsymbol{\beta^C_{ML}}$ estimator in eq.13, we need to show the $\hat{\beta}_{ML}$ estimator in eq.6 $\hat{\beta}_{ML} = \left(X^/\widehat{W}X\right)^{-1}\left(X^/\widehat{W}Z\right)$ since $\widehat{w}$is symmetric p.d matrix, there is found an orthogonal matrix F such that $\widehat{w} = F^TF$

$$L = (Z^* - x^*\hat{\beta}_{ML})^/(Z^* - x^*\hat{\beta}_{ML}) + \lambda(r - R\hat{\beta}_{ML})$$

$$\frac{\partial}{\partial \hat{\beta}_{ML}} L = -2x^{*/} Z^{*} + 2(x^{*/} x^{*})\beta_{ML}^{C} + R^{/}\lambda = 0$$

$$\frac{\partial}{\partial \lambda} L = r - R\beta_{ML}^{C} = 0$$

$$\lambda = -2(R(x^{*/} x^{*})^{-1}R^{/})^{-1}(r - R\hat{\beta}_{ML})$$

where $x^{*} = FX$, $Z^{*} = FZ$. If $\left(r - R\hat{\beta}_{ML}\right) = 0$, then $\beta_{ML}^{C} = \hat{\beta}_{ML}$

2.5 The MSE properties of the BCML estimator

The BCML was unbiased where $\mathbb{E}(\beta_{ML}^{C}) = \hat{\beta}_{ML}$, the mean square error is given by

$$MSE\left(\boldsymbol{\beta}_{\boldsymbol{ML}}^{\boldsymbol{C}}\right) = tr\left[\frac{1}{\varphi}\left(X^{/}\widehat{W}X + R^{/}R\right)^{-1}\right] \tag{14}$$

Lemma 2.5.1. The MSE for the constrained beta regression estimator is always less than the traditional one.

Proof. since $\frac{1}{\varphi}tr\left(X^{T}\widehat{W}X + R^{T}R\right) > \frac{1}{\varphi}tr(X^{T}\widehat{W}X)$, so $\frac{1}{\varphi}tr\left(X^{T}\widehat{W}X + R^{T}R\right)^{-1} <$ $\frac{1}{\varphi}tr\left(X^{T}\widehat{W}X\right)^{-1}$, then $MSE\left(\boldsymbol{\beta}_{\boldsymbol{ML}}^{\boldsymbol{C}}\right) < MSE\ (\widehat{\boldsymbol{\beta}}_{\boldsymbol{ML}})$

3. Real data application

The wellbeing index of Turkey 2015 has been used in this study; the index involves the variables that affect on level of happiness, Work, Education, and Environment, have been used in this study. As the level of happiness variable lies between zero and one. A better happiness is indicated as the values near to one. The Turkish Statistics Association has been used to obtain the data (Aktas¸ and Unlu 2017).

A model of beta regression was done where the dependent variable is the level of happiness. Data consists of 41 variables. Table 1 indicates only nine variables that were selected in this study. Table 1 introduces descriptive measures for the selected variables. Histogram, QQ, PP plot, and theoretical CDF have been used to check the goodness of fit measures that are used to fit the data to the beta distribution, Figure 1. indicates that plots show that the dependent variable follows the beta distribution.

The correlation matrix for the selected variables has been done between the independent variables, which indicates that all correlation values lie between (0.009 – 0.664), which means that no multicollinearity was found in the data. The parameter estimates for the model of beta regression using the ML method and constrained maximum likelihood method are indicated in Table 2. The constrained method assumes that there was found prior information about the estimators, which should be to equal one for its sum (the percentage variable). Eqs. (8) and (14), have been used to estimate the MSE of ML and BCML estimators respectively. The results indicate that MSE is reduced when we use additional information about estimators. The MSE for the constrained estimator is less than The MSE for the traditional one.

To judge on the constrained beta regression estimator, relative efficiency (RE) has been calculated as it equal $\frac{\text{MSE } ML}{\text{MSE BCML}} = \frac{5.259}{2.724} \sim 1.93$. This result means that the new method for estimating the estimators has half the MSE of the traditional method.

Table 1: Description of selected variables.

Field	Variable	Symbol	Mean	Standard deviation
-	Level of happiness (%)	y	0.612	0.075
Work	Employment rate (%)	x1	0.462	0.062
	Average daily earnings (TRY)	x2	0.577	0.066
	Job satisfaction rate (%)	x3	0.788	0.065
Education	Net schooling ratio of pre-primary education between the ages of 3 and 5(%)	x4	0.353	0.062
	Percentage of higher education graduates (%)	x5	0.131	0.023
	Satisfaction rate with public education services (%)	x6	0.741	0.085
Environment	Percentage of households having noise problems from the streets (%)	x7	0.157	0.059
	Satisfaction rate with municipal cleaning services (%)	x8	0.640	0.143

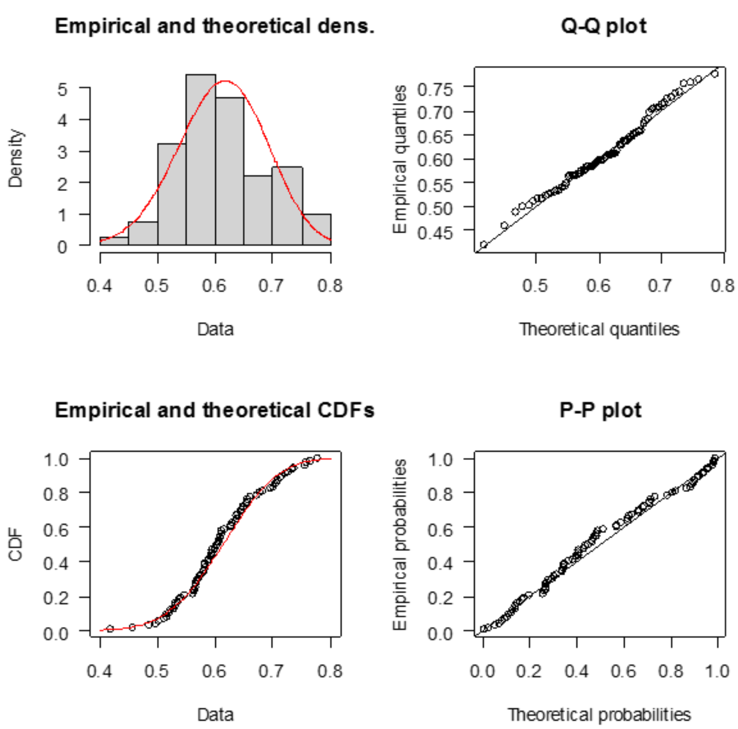

Figure 1: Fitting data to beta distribution

Table 2: Estimated MSE values and Parameters of ML and BCML estimators

Variable	ML	BCML
Intercept	0.488	-0.850
x1	0.460	-0.895
x2	0.452	-0.346
x3	0.721	3.261
x4	0.515	-1.179
x5	1.776	3.157
x6	0.495	0.462
x7	0.593	-1.875
x8	0.262	-1.082
MSE	5.259	2.724

4. Simulation study

A simulation study has been done in this section to show the performances of ML with the suggested constrained beta estimators BCML.

4.1 The methodology of the simulation

The simulation was done using both the model of beta regression, and the constrained model of beta regression that indicated in Eq. (2), and Eq. (13) respectively. The data were generated from a multivariate normal distribution with a fixed vector of $\mu = 0.3$, and different types of variance-covariance matrix.

The variance-covariance matrix is designed to contain different values of correlation (ρ) between the independent variables (0.2, 0.5, and 0.8). The number of the independent variables was 4 and 6. The Precision parameter was 2 and 4. In addition, different sample sizes were used 100, 250, and 500. MSE and relative efficiency (RE) were the criteria for the performance of ML and BCML. Simulation has been done using the following Steps:

1. Generate dependent variable that follow beta distribution.
2. Generate independent variables that follow multivariate normal distribution.
3. Set the prior information (matrix R) which lead to the summation of the coefficient must be equal 1.
4. Estimate traditional beta ($\hat{\beta}_{ML}$) and calculate its MSE (equations 6-8).
5. Estimate new beta (BCML) and calculate its MSE (equations 13-14).
6. Calculate relative efficiency = MSE in step (4) divided by MSE in step (5).
7. Replicate all previous steps 1000 times.
8. The average value of the 1000 has been used in the study.

4.2 Relative efficiency

Relative efficiency is used to compare performance between the two estimators. (Abonazel and Taha, 2021); it can be calculated as:

$$RE(\boldsymbol{\beta}_{ML}^{C}) = \frac{MSE\ (\widehat{\boldsymbol{\beta}}_{ML})}{MSE\ (\boldsymbol{\beta}_{ML}^{C})}$$

4.3 Simulation results

Tables 3–8 indicate the results of the simulation. It shows the MSE for each ML and BCML. It also indicates RE for the two estimators.

Precision parameter φ, the number of independent variables p, the degrees of correlation ρ, and the sample size n, are considered to be the factors that affect the MSE values of the ML and BCML estimators in the simulation

Table 3: Values of MSE at different cases of both ML and BCML estimators when $\varphi = 2$, and $p = 2$

Sample size	ML	BCML	RE
	$\boldsymbol{\rho = 0.2}$		
100	0.203	0.009	24.02
250	0.074	0.003	21.85
500	0.036	0.002	21.24
	$\boldsymbol{\rho = 0.5}$		
100	0.205	0.010	22.41
250	0.074	0.004	20.36
500	0.036	0.002	19.66
	$\boldsymbol{\rho = 0.8}$		
100	0.210	0.018	12.27
250	0.078	0.007	11.34
500	0.038	0.003	11.11

Table 4: Values of MSE at different cases of both ML and BCML estimators when $\varphi = 4$, and $p = 2$

Sample size	ML	BCML	RE
	$\boldsymbol{\rho = 0.2}$		
100	0.778	0.008	100.70
250	0.286	0.003	91.10
500	0.138	0.002	87.93
	$\boldsymbol{\rho = 0.5}$		
100	0.780	0.009	94.70
250	0.284	0.003	85.62
500	0.140	0.002	84.39
	$\boldsymbol{\rho = 0.8}$		
100	0.788	0.016	50.96
250	0.287	0.006	46.91
500	0.140	0.003	45.54

Table 5: Values of MSE at different cases of both ML and BCML estimators when $\varphi = 2$, and $p = 4$

Sample size	ML	BCML	RE
	$\boldsymbol{\rho = 0.2}$		
100	0.214	0.018	11.97
250	0.077	0.007	11.15
500	0.037	0.003	11.01
	$\boldsymbol{\rho = 0.5}$		
100	0.220	0.022	10.31
250	0.080	0.008	9.66
500	0.039	0.004	9.49
	$\boldsymbol{\rho = 0.8}$		
100	0.248	0.048	5.21
250	0.090	0.019	4.91
500	0.044	0.009	4.83

Table 6: Values of MSE at different cases of both ML and BCML estimators when $\varphi = 4$, and $p = 4$

Sample size	ML	BCML	RE
	$\boldsymbol{\rho = 0.2}$		
100	0.824	0.017	48.84
250	0.296	0.007	45.67
500	0.140	0.003	43.86
	$\boldsymbol{\rho = 0.5}$		
100	0.830	0.020	42.45
250	0.298	0.008	39.81
500	0.141	0.004	38.16
	$\boldsymbol{\rho = 0.8}$		
100	0.854	0.044	19.87
250	0.308	0.017	18.62
500	0.146	0.008	17.85

Figure 2 indicates the summary of RE values for 54 different cases. Figure 2 indicates the average of relative efficiency for the 1000 replications for the two estimators. It can be noticed that relative efficiency increases as the number of dependent variables decreases. It also indicates that the RE of all cases greater than one which mean (BCML better than traditional ML)

Table 7: Values of MSE at different cases of both ML and BCML estimators when $\varphi = 2$, and $p = 6$

Sample size	ML	BCML	RE
$\rho = 0.2$			
100	0.231	0.028	8.56
250	0.0814	0.010	7.95
500	0.039	0.005	7.74
$\rho = 0.5$			
100	0.242	0.035	7.07
250	0.085	0.013	6.57
500	0.041	0.006	6.40
$\rho = 0.8$			
100	0.290	0.081	3.63
250	0.103	0.030	3.41
500	0.050	0.015	3.33

Table 8: Values of MSE at different cases of both ML and BCML estimators when $\varphi = 4$, and $p = 6$

Sample size	ML	BCML	RE
$\rho = 0.2$			
100	0.853	0.026	33.90
250	0.293	0.010	30.47
500	0.143	0.005	30.50
$\rho = 0.5$			
100	0.863	0.032	27.63
250	0.297	0.012	25.03
500	0.145	0.006	25.02
$\rho = 0.8$			
100	0.906	0.073	12.54
250	0.313	0.028	11.40
500	0.152	0.013	11.37

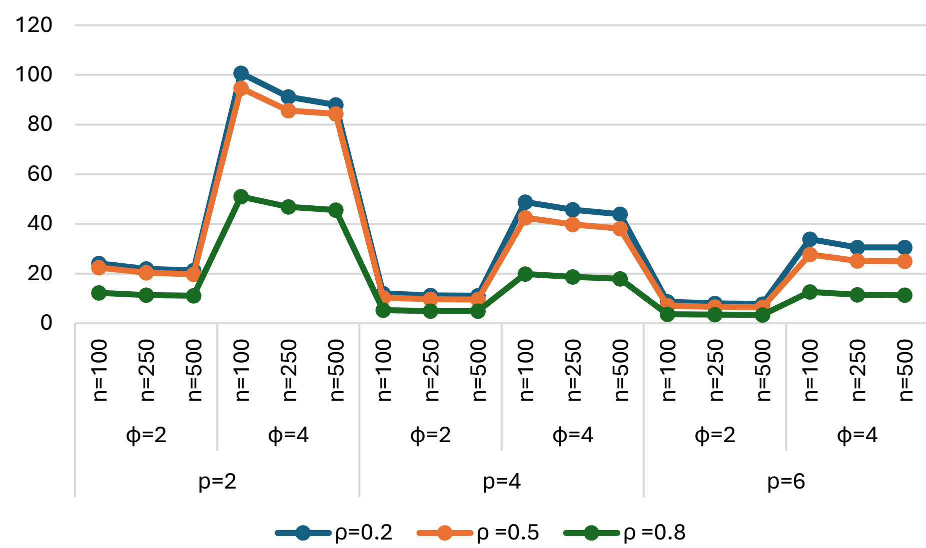

Figure 2: Relative efficiency for different cases.

5. Conclusions

Prior Information analyzes the data better. Ignoring this prior information may affect the decision made by researchers. Therefore, the constrained beta estimator has been introduced in this paper to get more benefit estimators because of all prior information about estimators has been used. The constrained beta regression estimator is better than the traditional one in the sense of MSE criteria.

A simulation experiment with different factors has been done. The results also indicate that the constrained beta regression estimator has a lower MSE in all cases, which lead to that all values of relative efficiency greater than one (MSE for BCML < the traditional estimator that evaluated by maximum likelihood method) this means that the constrained beta regression estimator is more useful than the maximum likelihood beta regression.

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