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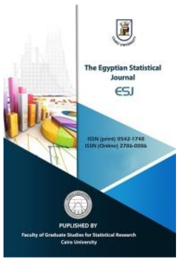

Some Characterizations Based on Generalized Order Statistics from Weibull-Weibull Distribution

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ABSTRACT

Characterizations of distributions based on recurrence relations for single and product moments of generalized order statistics have been investigated quite extensively in the literature involving ordered random variables. As generalized order statistics (GOS) provide a unifying approach to models of ordered random variables, we establish here some characterizations on absolutely continuous distributions based on GOS which contain and strengthen several known results in this regard. Because we do not impose restrictions on the model parameters (as done in the most of previous studies), our findings yield new results for various useful models of ordered random variables including k-record values, sequential order statistics, and progressively Type-II censored order statistics with an arbitrary censoring plan.

The present paper is devoted to derive some recurrence relations for single and product moments of generalized order statistics for Weibull - Weibull distribution (WWD). Based on these recurrence relations, some characterizations for this distribution are discussed.

Keywords:

Generalized order statistics – Records – Recurrence relations - Weibull- Weibull distribution - Characterization.

1. Introduction

Characterizations of distributions based on ordered random variables have received considerable attention in the literature. Among this, Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Ahsanullah (2016), Pawlas and Szynal (2001), Ahmed (2007), Ahmed and Fawzy (2007), Khan et al. (2007), AL-Hussaini et al. (2005) and Kumar (2011). Abdul-Moniem (2019), Nagwa (2020), Alimohammadi (2022), some of them discussed characterizations by conditional events of generalized order statistics.

The aim of the present article is to provide some characterizations for absolutely continuous distributions based on recurrence relations for single and product moments of GOS. In our study, we do not want to extend all characterization results in this regard. But, our findings and mathematical methods not only yield new characterization results for various useful models of ordered random variables but also could be used in different aspects of GOS.

An interesting method of adding a new parameter to an existing G distribution has been proposed by Bourguignon et al. (2014). The resulting distribution, known as the Weibull generated distribution, includes the original distribution as a special case and gives more flexibility to model various types of data.

Let $G(x,\xi)$ be a continuous baseline distribution with density $g(x,\xi)$ depends on a parameter vector ξ and the following cumulative distribution function (*cdf*) of Weibull

$$F(x,\alpha,\beta)=1-e^{-\alpha x^{\beta}}; \qquad x>0,\alpha,\beta>0.$$

The *cdf* of the Weibull– G family is given by

$$F(x,\alpha,\beta,\xi)=\int_0^{\left(\frac{G(x)}{\bar{G}(x)}\right)} \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx=1-e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}.$$

The reliability function of the Weibull– G family is given by

$$\bar{F}(x)=e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}; \qquad x\geq 0,\beta,\alpha>0, \tag{1}$$

where α and β are the scale and shape parameters, respectively. The probability density function (*pdf*) corresponding to $\bar{F}(x)$ is:

$$f(x)=\alpha\beta g(x)\frac{G(x)^{\beta-1}}{\bar{G}(x)^{\beta+1}}e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}}; \quad x\geq 0,\beta,\alpha>0, \tag{2}$$

Here, $\bar{G}(x)$ is denoted as

$$\bar{G}(x)=\mathrm{e}^{-\lambda x^{\theta}}; \qquad x\geq 0,\ \lambda,\theta>0, \tag{3}$$

where θ and λ are the scale and shape parameters. Substituting from Eq.(3) in Eq.(1), we get

$$\bar{F}(\mathrm{x})=\mathrm{e}^{-\alpha\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}; \qquad x\geq 0 \tag{4}$$

The *pdf* corresponding to $\bar{F}(x)$ will be as

$$f(x)=\alpha\theta\beta\lambda x^{\theta-1}e^{\lambda x^{\theta}}\left(e^{\lambda x^{\theta}}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}; \quad x\geq 0. \tag{5}$$

The distribution in Eq.(5), is called Weibull-Weibull distribution (WWD) as in Bourguignon et al. (2014).

Now in view of Eq.(4) and Eq.(5), we get

$$\bar{F}(x)=\frac{f(x)}{\alpha\theta\beta\lambda}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}\lambda^{v}x^{1-\theta(1-v)}}{v!}. \tag{6}$$

Then distributions can be obtained from Eq.(5), as showed.

Table 1: Sub Models

θ	***pdf***	**Distribution**
1	$\alpha\beta\lambda e^{\lambda x}\left(e^{\lambda x}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda x}-1\right)^{\beta}} \quad x\geq 0$	Weibull- exponential
2	$\alpha\theta\beta\lambda x e^{\lambda x^{2}}\left(e^{\lambda x^{2}}-1\right)^{\beta-1}\mathrm{e}^{-\alpha\left(e^{\lambda x^{2}}-1\right)^{\beta}} \quad x\geq 0$	Weibull- Rayleigh

The concept of GOS was introduced by Kamps (1995). A variety of order models of random variables is contained in this concept. Let, for simplicity *F*, throughout denote an absolutely continuous distribution function with density function *f*.

The random variables $X(1,n,\tilde{m},k),\ldots, X(n,n,\tilde{m},k)$ are called generalized order statistics based on *F*, if their joint *pdf* of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i} f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1} f(x_n),$$

for $F^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$. with parameters $n \in N,\ k \ge 2,\ k > 0,$ $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1}, M_r = \sum_{i=r}^{n-1} m_i$, such that $\gamma_r = k + n - r + M_r > 0,$ for all $r \in \{1,2,\ldots,n-1\}$. For $\gamma_i \ne \gamma_j$ for all $i, j \in (1,2,\ldots,n-1)$ the pdf of $X(r,n,\tilde{m},k)$ is given by Cramer and Kamps (2000) in the following way

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r)\left[\bar{F}(x)\right]^{\gamma_i - 1}. \tag{7}$$

The joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is given as

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = C_{s-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\left(\sum_{i=1}^{r} a_i^{(r)}\left[\bar{F}(x)\right]^{\gamma_i}\right)\frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)}, \quad x < y, \tag{8}$$

where

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \le i \le n,$$

and

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \le i \le s \le n.$$

It may be noted that for $m_1 = m_2 = \ldots = m_{n-1} = m \ne -1$

$$a_i(r) = \frac{(-1)^{r-i}\binom{r-1}{r-i}}{(m+1)^{r-1}(r-1)!}, \tag{9}$$

and

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}\binom{s-r-1}{s-i}}{(m+1)^{s-r-1}(s-r-1)!}.$$

(10)

Therefore *pdf* of $X(r,n,\tilde{m},k)$ given in Eq.(7) reduced to

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{\Gamma(r)}\left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left[F(x)\right], \quad x \in \chi, \tag{11}$$

and joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k), 1 \le r < s \le n$ is given in Eq.(8) reduced to

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\Gamma(s-r-1)!}\left[\bar{F}(x)\right]^{m} f(x) g_m^{r-1}\left[F(x)\right]$$
$$\left\{h_m\left[F(y)\right] - h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s - 1} f(y), \quad x < y, \tag{12}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\quad \gamma_i=k+(n-i)(m+1),$$

$$h_m(x)=\begin{cases}\dfrac{-(1-x)^{m+1}}{m+1}, & m=-1\\ \log\left(\dfrac{1}{1-x}\right), & m\neq -1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1),\quad x\in[0,\ 1).$$

We shall also take $X(0,n,m,k)=0$. If m = 0, k = 1, then $X(r,n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1,X_2,...,X_n$ and when m = -1, then $X(r,n,m,k)$ reduces to the k^{th} record value (Pawlas and Szynal (2001)).
The r^{th} generalized TL-moments with t_1 smallest and t_2 largest trimming are defined as follows

$$L_r^{(t_1,t_2)}=\frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left(X_{r-k+t_1:r+t_1+t_2}\right);\quad t_1,t_2=1,2,\ldots \text{ and } r=1,2,\ldots, \tag{13}$$

where $E\left(X_{r-i+t_1:r+t_1+t_2}\right)$ is the expected value of the $(r-i+t_1)^{th}$ order statistics of the random sample of size $(r+t_1+t_2)$. The case $t_1=t_2=0$ yields the original L-moments defined by Hosking (1990). These relations are obtained in the following sections.

2. Recurrence relation for single Expectations of GOS

In this section, the single moments of GOS for WWD are obtained. Moments of order statistics, TL-moments and L- moments are obtained as a special case of single moments of GOS. Recurrence relations for single moments of GOS are also provided.

The single moments of GOS for WWD are

$$\begin{aligned}E\left[X^j(r,n,\tilde{m},k)\right]&=\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx\\&=\frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)\left[1-\left(\bar{F}(x)\right)^{m+1}\right]^{r-1}dx\\&=\frac{C_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r+w(m+1)-1}f(x)dx\\&=\frac{jC_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!\left[\gamma_r+w(m+1)\right]}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r+w(m+1)}dx.\end{aligned}$$

Using Eq.(4), we get

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]=\frac{jC_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}\left(-1\right)^{w}}{\left(m+1\right)^{r-1}\left(r-1\right)!\left[\gamma_{r}+w\left(m+1\right)\right]}\int_{0}^{\infty}x^{j-1}\left[\mathrm{e}^{-\alpha\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}\right]^{\gamma_{r}+w(m+1)}dt. \quad (14)$$

let $a=\alpha\left[\gamma_{r}+w\left(m+1\right)\right]$.

$$I_{1}=\int_{0}^{\infty}x^{j-1}\left[\mathrm{e}^{-\alpha\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}\right]^{\gamma_{r}+w(m+1)}dx=\int_{0}^{\infty}x^{j-1}\,\mathrm{e}^{-a\left(e^{\lambda x^{\theta}}-1\right)^{\beta}}dx\,.$$

First, to obtain I_1, the binomial expansion is employed as follows

$$I_{1}=\int_{0}^{\infty}x^{j-1}\sum_{\delta=0}^{\infty}\frac{\left(-1\right)^{\delta}a^{\delta}\left(e^{\lambda x^{\theta}}-1\right)^{\beta\delta}}{\delta!}dx=\sum_{\delta=0}^{\infty}\frac{\left(-1\right)^{\delta}a^{\delta}}{\delta!}\int_{0}^{\infty}x^{j-1}\left(e^{\lambda x^{\theta}}-1\right)^{\beta\delta}dx\,, \quad (15)$$

$$I_{1}=\sum_{\delta=0}^{\infty}\frac{\left(-1\right)^{\delta}a^{\delta}}{\delta!}\int_{0}^{\infty}x^{j-1}\left(e^{\lambda x^{\theta}}-1\right)^{\beta\delta}dx=\sum_{\delta=0}^{\infty}\frac{\left(-1\right)^{\delta}a^{\delta}}{\delta!}\int_{0}^{\infty}x^{j-1}e^{\lambda\beta\delta x^{\theta}}\left(1-e^{-\lambda x^{\theta}}\right)^{\beta\delta}dx\,.$$

Again, the binomial expansion is employed in Eq.(14) as the following

$$I_{1}=\sum_{\delta=0}^{\infty}\frac{\left(-1\right)^{\delta}a^{\delta}}{\delta!}\int_{0}^{\infty}x^{j-1}e^{\lambda\beta\delta x^{\theta}}\left(1-e^{-\lambda x^{\theta}}\right)^{\beta\delta}dx=\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{\beta\delta}{\eta}\frac{\left(-1\right)^{\eta+\delta}a^{\delta}}{\delta!}\int_{0}^{\infty}x^{j-1}e^{-\eta\lambda x^{\theta}}dx,$$

let $y=\eta\lambda x^{\theta}\Rightarrow x=\left(\eta\lambda\right)^{\frac{-1}{\theta}}y^{\frac{1}{\theta}}\Rightarrow dx=\left(\eta\lambda\right)^{\frac{-1}{\theta}}\frac{1}{\theta}y^{\frac{1}{\theta}-1}dy,$

$$I_{1}=\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{\beta\delta}{\eta}\frac{\left(-1\right)^{\eta+\delta}a^{\delta}}{\delta!}\int_{0}^{\infty}\left(\left(\eta\lambda\right)^{\frac{-1}{\theta}}y^{\frac{1}{\theta}}\right)^{j-1}e^{-y}\left(\eta\lambda\right)^{\frac{-1}{\theta}}\frac{1}{\theta}y^{\frac{1}{\theta}-1}dy.$$

So, I_1 is given by

$$I_{1}=\frac{1}{\theta}\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{\beta\delta}{\eta}\frac{\left(-1\right)^{\eta+\delta}\left(\eta\lambda\right)^{\frac{-j}{\theta}}a^{\delta}}{\delta!}\Gamma\left(\frac{j}{\theta}\right).$$

From Eq.(14), The single moments of GOS for WWD will be

$$E\left[T^{j}\left(r,n,\tilde{m},k\right)\right]=\frac{jC_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{\beta\delta}{\eta}\frac{\left(-1\right)^{w+\eta+\delta}\left(\eta\lambda\right)^{\frac{-j}{\theta}}a^{\delta}}{\delta!}\Gamma\left(\frac{j}{\theta}\right)}{\theta\left(m+1\right)^{r-1}\left(r-1\right)!\left[\gamma_{r}+w\left(m+1\right)\right]}. \quad (16)$$

which is the expression of single moments of GOS from the WWD.

2.1 Moments of Upper Order Statistics

In this subsection, the single moments of GOS for WWD are obtained based on Eq.(16). Also, numerical values of the mean and variance of upper order statistics for some choices values of parameters are calculated.

The j^{th} moment of upper order statistics is obtained by taking *m*=0, *k*=1 in Eq.(16) as follows

$$E\left(T_{n-r+1:n}^{j}\right)=\frac{jn!\sum_{w=0}^{r-1}\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{r-1}{w}\binom{\beta\delta}{\eta}\left(-1\right)^{w+\eta+\delta}\left(\eta\lambda\right)^{\frac{-j}{\theta}}\left(n-r+w\right)^{\delta}\alpha^{\delta}}{\theta(n-r)!\left(r-1\right)!\delta!\left(n-r+w\right)}\Gamma\left(\frac{j}{\theta}\right).$$

Or, by substituting $n-r+1=r$, the $E\left(T_{n-r+1:n}^{j}\right)$ will be $E\left(T_{r:n}^{j}\right)$ and takes the following form

$$E\left(T_{r:n}^{j}\right)=\frac{jn!\sum_{w=0}^{r-1}\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{n-r}{w}\binom{\beta\delta}{\eta}(-1)^{w+\eta+\delta}(\eta\lambda)^{\frac{-j}{\theta}}(n-r+w)^{\delta}\alpha^{\delta}}{\theta(n-r)!(n-r)!\delta!(n-r+w)}\Gamma\left(\frac{j}{\theta}\right). \tag{17}$$

Some values of mean and variance of order statistics for the WWD are calculated for some values of parameters in Tables 2 and 3.

Table 2: Mean of order statistics for WWD

n	r	$\alpha=0.3,\ \beta=0.5,\ \lambda=0.4$	$\alpha=1,\ \beta=0.3,\ \lambda=0.4$	$\alpha=0.5,\ \beta=0.3,\ \lambda=1$	$\alpha=0.5,\ \beta=0.8,\ \lambda=1$
1	1	38.597	279.001	71.196	1.105
2	1	15.431	27.745	13.156	0.595
	2	61.762	530.258	129.235	1.615
3	1	7.992	4.787	3.741	0.392
	2	30.309	73.66	31.987	1
	3	77.489	758.557	177.859	1.922
4	1	4.692	1.089	1.741	0.285
	2	17.891	15.88	11.04	0.714
	3	42.727	131.441	52.934	1.286
	4	89.076	967.596	219.5	2.135
5	1	2.977	0.296	0.52	0.219
	2	11.552	4.262	4.463	0.547
	3	27.4	33.308	20.906	0.965
	4	52.946	196.863	74.285	1.499
	5	98.109	0.00116	255.804	2.293
6	1	1.994	0.091	0.226	0.175
	2	7.893	1.318	1.99	0.437
	3	18.872	10.148	9.408	0.766
	4	35.928	56.468	32.405	1.164
	5	61.455	267.06	95.226	1.667
	6	105.439	0.001339	287.92	2.419
7	1	1.391	0.031	0.105	0.144
	2	5.613	0.452	0.951	0.361
	3	13.592	3.484	4.587	0.63
	4	25.911	19.033	15.836	0.948
	5	43.44	84.545	44.831	1.327
	6	68.66	340.066	115.383	1.803
	7	111.569	0.001505	316.676	2.521
8	1	1.002	0.012	0.051	0.121
	2	4.116	0.169	0.479	0.304
	3	10.105	1.303	2.366	0.53
	4	19.403	7.118	8.288	0.795
	5	32.42	30.948	23.384	1.101
	6	50.053	116.702	57.7	1.462
	7	74.864	414.52	134.611	1.917
	8	116.813	0.001661	342.685	2.608

Note that: the results in Table 2 are consistent with property of order statistics $\sum_{i=1}^{n} \mu_{i:n} = n\mu_{1:1}$ given by David and Nagaraja (2003).

For example: based on Table 2.

$$\sum_{i=1}^{2} \mu_{1:2} = 15.431 + 61.762 = 77.193,$$

and,

$$2\mu_{1:1} = 2 \times 38.597 = 77.193,$$

then $\sum_{i=1}^{2} \mu_{1:2} = 2\mu_{1:1}$, which justify this property.

Table 3: Variance of order statistics for WWD

n	r	$\alpha=0.3,\ \beta=0.5,\ \lambda=0.4$	$\alpha=1,\ \beta=0.3,\ \lambda=0.4$	$\alpha=0.5,\ \beta=0.3,\ \lambda=1$	$\alpha=0.5,\ \beta=0.8,\ \lambda=1$
1	1	0.002046	157.567	0.0002581	0.854
2	1	550.461	17.359	0.002053	0.345
	2	0.002469	246.608	0.0004284	3.842
3	1	208.733	3.431	318.569	0.18
	2	901.881	39.175	0.00499	0.428
	3	0.00251	301.001	0.0005467	0.766
4	1	93.86	0.918	68.052	0.107
	2	422.695	9.805	999.091	0.26
	3	0.001073	59.94	0.008104	0.432
	4	0.002452	336.105	0.0006326	0.697
5	1	25.351	0.298	17.681	0.07
	2	222.576	3.098	257.1	0.173
	3	572.191	17.708	0.00195	0.286
	4	0.001145	78.024	0.0001107	0.416
	5	0.002371	359.681	0.0006972	0.642
6	1	25.351	0.111	5.259	0.048
	2	126.095	1.14	77.196	0.121
	3	335.186	6.346	580.223	0.203
	4	663.736	26.044	0.003055	0.29
	5	0.00169	93.254	0.0001376	0.394
	6	0.002289	375.979	0.0007472	0.597
7	1	14.483	0.046	1.73	0.034
	2	75.278	0.469	25.821	0.089
	3	207.672	2.579	196.191	0.151
	4	418.478	10.295	0.00102	0.215
	5	715.996	34.168	0.004221	0.284
	6	0.001168	105.925	0.0001615	0.373
	7	0.002213	387.481	0.0007869	0.561
8	1	8.655	0.021	0.615	0.025
	2	46.791	0.21	9.37	0.067
	3	133.841	1.149	72.504	0.116
	4	276.689	4.533	380.421	0.166
	5	475.551	14.601	0.001546	0.218
	6	743.666	41.755	0.005385	0.275
	7	0.001156	116.441	0.0001826	0.354
	8	0.002144	395.708	0.0008191	0.53

2.2 TL Moments

In this subsection, the r^{th} TL- moment and r^{th} L- moment for the WWD are obtained.

The r^{th} TL- moment can be obtained from Eq.(13) and Eq.(17) with $j=1$, $n=r+t_1+t_2$ and $n-r+1=r-k+t_1$ as follows:

$$E\left(T_{r-k+t_1:r+t_1+t_2}\right)=\sum_{w=0}^{n-r+k-t_1}\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{n-r+k-t_1}{w}\binom{\beta\delta}{\eta}$$
$$\frac{(-1)^{w+\eta+\delta}(\eta\lambda)^{\frac{-1}{\theta}}(r+t_1+t_2)!(n-r+w)^{\delta}\alpha^{\delta}}{\theta(n-r+k-t_1)!(r-k+t_1-1)!\delta!(n-r+w)}\Gamma\left(\frac{1}{\theta}\right).$$

Then, the r^{th} TL- moment of the WWD is obtained by substituting the previous expectation in Eq.(13) as follows.

Furthermore, the r^{th} L- moments can be obtained from Eq.(17) with $t_1=t_2=0$ as follows:

$$L_r^{(t_1,t_2)}=\frac{(\eta\lambda)^{\frac{-1}{\theta}}}{\theta r}\sum_{k=0}^{r-1}\frac{(r+t_1+t_2)!(-1)^k\binom{r-1}{k}}{(n-r+k-t_1)!(r-k+t_1-1)!}\sum_{w=0}^{n-r+k-t_1}\sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty}\binom{n-r+k-t_1}{w}$$
$$\binom{\beta\delta}{\eta}\frac{(-1)^{w+\eta+\delta}(r+t_1+t_2)!(n-r+w)^{\delta}\alpha^{\delta}}{(n-r+k-t_1)!(r-k+t_1-1)!\delta!(n-r+w)}\Gamma\left(\frac{1}{\theta}\right);\quad t_1,t_2=1,2,\ldots \tag{18}$$

and $r=1,2,\ldots$

Furthermore, the r^{th} L- moments can be obtained from Eq.(18) with $t_1=t_2=0$ as follows:

$$L_r=\theta^r\sum_{k=0}^{r-1}\frac{r!(-1)^k\binom{r-1}{k}}{(n-r+k)!(r-k-1)!}\sum_{w=0}^{n-r+k}\sum_{\delta=0}^{\infty}\binom{n-r+k}{w}$$
$$\frac{(-1)^{w+\delta}r!\alpha^{\delta}\left[\gamma_r+w(m+1)\right]^{\delta-1}}{\delta!(n-r+k)!(r-1)!}\frac{\Gamma(\beta\delta+j)\Gamma(1-\beta\delta)}{\Gamma(\beta\delta+j+1)}. \tag{19}$$

The first four L-moments can be obtained from Eq.(17) by taking $r=1,2,3$ and 4 respectively. Using Eq.(18), some numerical results for $L_1^{(t_1,t_2)}, L_2^{(t_1,t_2)}, L_3^{(t_1,t_2)}, L_4^{(t_1,t_2)}, L_1, L_2, L_3, L_4, \tau_1^{(t_1,t_2)}, \tau_3^{(t_1,t_2)}, \tau_4^{(t_1,t_2)}, \tau_1, \tau_2$ and τ_3 are obtained in Table 4.

Using Eq.(17), some numerical results for mean and variance of order statistics are obtained in Athar and Islam (2004).

Table 4: Some numerical results for the r^{th} L- moments for different values of parameters

(t_1,t_2)		$(1,1)$	$(2,2)$	$(0,1)$	$(0,2)$	$(1,0)$	$(2,0)$	$(0,0)$
$\alpha=0.3$, $\beta=0.5$, $\lambda=0.4$	$L_1^{(t_1,t_2)}$	30.31	27.4	15.43	7.99	61.76	77.49	38.597
	$L_2^{(t_1,t_2)}$	12.42	8.53	11.16	6.59	23.59	23.174	23.166
	$L_3^{(t_1,t_2)}$	3.233	1.736	3.879	2.424	7.171	6.539	8.287
	$L_4^{(t_1,t_2)}$	0.598	0.224	0.607	0.249	2.48	2.497	2.469
	$\tau_1^{(t_1,t_2)}$	3.213						
	$\tau_3^{(t_1,t_2)}$	0.204						
	$\tau_4^{(t_1,t_2)}$	0.026						
$\alpha=1$, $\beta=0.3$, $\lambda=0.4$	$L_1^{(t_1,t_2)}$	3.569	2.633	1.562	0.559	11.679	15.733	6.621
	$L_2^{(t_1,t_2)}$	2.074	1.23	1.505	0.624	6.082	6.727	5.058
	$L_3^{(t_1,t_2)}$	1.04	0.521	0.967	0.429	3.102	3.097	3.052
	$L_4^{(t_1,t_2)}$	0.441	0.185	0.459	0.198	1.542	1.498	1.601
	$\tau_1^{(t_1,t_2)}$	2.141						
	$\tau_3^{(t_1,t_2)}$	0.423						
	$\tau_4^{(t_1,t_2)}$	0.151						
$\alpha=0.5$, $\beta=0.3$, $\lambda=1$	$L_1^{(t_1,t_2)}$	31.99	20.91	13.16	3.74	129.24	177.86	71.19
	$L_2^{(t_1,t_2)}$	20.95	11.49	14.12	4.87	72.94	83.28	58.04
	$L_3^{(t_1,t_2)}$	12.31	5.92	10.72	4.17	41.56	42.71	39.21
	$L_4^{(t_1,t_2)}$	6.06	2.51	6.11	2.48	22.80	22.51	23.13
	$\tau_1^{(t_1,t_2)}$	1.818						
	$\tau_3^{(t_1,t_2)}$	0.514						
	$\tau_4^{(t_1,t_2)}$	0.218						
$\alpha=0.5$, $\beta=0.8$, $\lambda=1$	$L_1^{(t_1,t_2)}$	1	0.965	0.595	0.392	1.615	1.922	1.105
	$L_2^{(t_1,t_2)}$	0.286	0.199	0.304	0.215	0.461	0.424	0.51
	$L_3^{(t_1,t_2)}$	0.039	0.02	0.047	0.03	0.092	0.087	0.105
	$L_4^{(t_1,t_2)}$	0.008	0.004	0.006	0.0008	0.036	0.036	0.034
	$\tau_1^{(t_1,t_2)}$	4.848						
	$\tau_3^{(t_1,t_2)}$	0.101						
	$\tau_4^{(t_1,t_2)}$	0.018						

3.Characterization based on recurrence relation for single moments of GOS

Theorem 3.1 Let *X* be a non-negative random variable having an absolutely continuous distribution function *F*(x) with *F*(0) = 0 and 0 < *F*(x) < 1 for all x > 0, then

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=\frac{j}{\alpha\beta\theta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}\lambda^{v-1}}{v!}E\left[X^{j-\theta(1-v)}\left(r,n,\tilde{m},k\right)\right]. \tag{20}$$

if and only if $\bar{F}(y)=e^{-\alpha\left(e^{\lambda y^{\theta}}-1\right)^{\beta}}$.

Proof

(i) The necessary proof

We have from Lemma 2.3 (see, Athar and Islam (2004)) that

$$E\left[\xi\left\{X\left(r,n,\tilde{m},k\right)\right\}\right]-E\left[\xi\left\{X\left(r-1,n,\tilde{m},k\right)\right\}\right]=C_{r-2}\int_{\theta}^{\beta}\xi'(x)\sum_{i=1}^{r}a_{i}(x)\left[\bar{F}(x)\right]^{\gamma_{r}}dx.$$

If we let $\xi(x)=x^{j}$, then

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=jC_{r-2}\int_{\theta}^{\beta}x^{j-1}\sum_{i=1}^{r}a_{i}(x)\left[\bar{F}(x)\right]^{\gamma_{r}}dx. \tag{21}$$

By substituting Eq.(6) in Eq.(21), we get

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=\frac{jC_{r-1}}{\alpha\beta\theta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}\lambda^{v-1}}{v!}\int_{0}^{\infty}x^{j-\theta(1-v)}\sum_{i=1}^{r}a_{i}(x)\left[\bar{F}(x)\right]^{\gamma_{r}-1}f(x)dx.$$

Which after simplification leads to Eq.(20).

(ii) The sufficient part

On the other hand if the recurrence relation in equation Eq.(20) is satisfied, then by using Eq.(13), we have

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=\frac{j}{\alpha\beta\theta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}\lambda^{v-1}}{v!}E\left[X^{j-\theta(1-v)}\left(r,n,\tilde{m},k\right)\right]$$

$$\frac{C_{r-1}}{(r-1)!}\int_{0}^{\infty}x^{j}\left[\bar{F}(x)\right]^{\gamma_{r}-1}f(x)g_{m}^{r-1}\left[F(x)\right]dx-\frac{C_{r-2}}{(r-2)!}\int_{0}^{\infty}x^{j}\left[\bar{F}(x)\right]^{\gamma_{r-1}-1}f(x)g_{m}^{r-2}\left[F(x)\right]dx,$$

$$=\frac{jC_{r-1}}{\alpha\beta\theta(r-1)!\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}\lambda^{v-1}}{v!}\int_{0}^{\infty}x^{j-\theta(1-v)}\left[\bar{F}(x)\right]^{\gamma_{r}-1}f(x)g_{m}^{r-1}\left[F(x)\right]dx.$$

Integrating the first term in the left hand side by parts, the expression will be

$$\frac{jC_{r-1}}{\gamma_{r}(r-1)!}\int_{0}^{\infty}x^{j-1}\left[\bar{F}(x)\right]^{\gamma_{r}}g_{m}^{r-1}\left[F(x)\right]dx$$

$$=\frac{jC_{r-1}}{\alpha\beta\theta(r-1)!\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}\left(\beta+u\right)^{v}\lambda^{v-1}}{v!}\int_{0}^{\infty}x^{j-\theta(1-v)}\left[\bar{F}(x)\right]^{\gamma_{r}-1}f(x)g_{m}^{r-1}\left[F(x)\right]dx$$

Therefore

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]$$
$$\left\{\bar{F}(x)-\frac{1}{\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v\lambda^{v-1}}{v!}x^{1-\theta(1-v)}f(x)\right\}dx=0. \tag{22}$$

Now applying a generalization of the Muntz-Szasz theorem (see, Hwang and Lin (1984)) to Eq.(22), we get

$$\bar{F}(x)-\frac{1}{\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v\lambda^{v-1}}{v!}x^{1-\theta(1-v)}f(x)=0,$$

Hence,

$$\bar{F}(x)=\frac{1}{\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v\lambda^{v-1}}{v!}x^{1-\theta(1-v)}f(x)$$
$$=\sum_{u=0}^{\infty}\binom{1-\beta}{u}(-1)^u\frac{e^{-(\beta+u)\lambda x^\theta}}{\alpha\beta\lambda\theta x^{\theta-1}}f(x).$$

Therefore,

$$\bar{F}(x)=\frac{e^{-\lambda x^\theta}f(x)}{\alpha\beta\theta\lambda x^{\theta-1}}\left(e^{\lambda x^\theta}-1\right)^{1-\beta}.$$

Integrating both sides from 0 to y, the equation will be as follows

$$\int_0^y\frac{f(x)}{\bar{F}(x)}dx=\alpha\beta\theta\lambda\int_0^y x^{\theta-1}e^{\lambda x^\theta}\left(e^{\lambda x^\theta}-1\right)^{\beta-1}dx.$$

This is implies that

$$-\ln\left[\bar{F}(y)\right]=\alpha\left(e^{\lambda y^\theta}-1\right)^\beta,$$

where $\bar{F}(y)=\mathrm{e}^{-\alpha\left(e^{\lambda y^\theta}-1\right)^\beta}$; $\quad y\geq 0$.

Corollary 3.2. For $m_1=m_2=\ldots=m_{n-1}=m\neq -1$, the recurrence relations for single moment of GOS for Weibull- Weibull distribution is given as

$$E\left[X^j(r,n,m,k)\right]-E\left[X^j(r-1,n,m,k)\right]=\frac{j}{\alpha\beta\theta\gamma_r}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}E\left[X^{j-\theta(1-v)}\lambda^{v-1}(r,n,m,k)\right]. \tag{23}$$

Proof. This can easy be deduced from Eq.(20) in view of the relation in Eq.(9).

Remark 3.1 By putting $m=0,\ k=1$ in Theorem 2.1., the recurrence relations for single moments of order statistics are obtained as

$$E\left(X_{r:n}^j\right)-E\left(X_{r-1:n}^j\right)=\frac{j}{\alpha\beta\theta(n-r+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}E\left(X_{r:n}^{j-\theta(1-v)}\lambda^{v-1}\right) \tag{24}$$

Remark 3.2 By setting $m=-1,\ k=1$ in Theorem 2.1., the recurrence relations of upper record values are obtained as

$$E\left[X^j(r,n,-1,1)\right]-E\left[X^j(r-1,n,-1,1)\right]$$
$$=\frac{j}{k\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^v}{v!}E\left[X^{j-\theta(1-v)}\lambda^{v-1}(r,n,-1,1)\right]. \tag{25}$$

4. Characterization based on recurrence relation for product moments of GOS

Theorem 4.1 Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $xy > 0$, then

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]$$
$$=\frac{j}{\alpha\beta\theta\gamma_{r}}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(\beta+u)^{v}\lambda^{v-1}}{v!}E\left[X^{i}(r,n,m,k)X^{j-\theta(1-v)}(s,n,m,k)\right]. \quad (26)$$

if and only if. $\bar{F}(y)=e^{-\alpha\left(e^{\lambda y^{\theta}}-1\right)^{\beta}}$.

Proof

(i) The necessary part

From Lemma 3.2 (see, Athar and Islam [2004]), it can be shown that

$$E\left[\xi\left\{X(r,n,m,k).X(s,n,m,k)\right\}\right]-E\left[\xi\left\{X(r,n,m,k).X(s-1,n,m,k)\right\}\right]=$$
$$\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]$$
$$\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}}dydx.$$

where $\xi(x,y)=\xi_{1}(x)\xi_{2}(y)$.

If we let $\xi(x,y)=x^{i}y^{j}$, then

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]=$$
$$\frac{C_{s-2}}{(r-1)!(s-r-1)!}\int_{\theta}^{\beta}\int_{x}^{\beta}x^{i}y^{j-1}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]$$
$$\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}}dydx.$$

On using Eq.(6) , we get

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]-E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]=$$
$$\frac{C_{s-2}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}}{\alpha\theta\beta\lambda(r-1)!(s-r-1)!v!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j-\theta(1-v)}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]$$
$$\left[h_{m}(F(y))-h_{m}(F(x))\right]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx.$$

Which after simplification leads to Eq.(26).

(ii) The sufficient part

If the recurrence relation in Eq.(26) is satisfied, then by using Eq.(12), we have

$$\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right]\left\{h_{m}\left[F(y)\right]-h_{m}\left[F(x)\right]\right\}^{s-r-1}$$
$$\left[\bar{F}(y)\right]^{\gamma_{s}-1}f(y)dydx-\frac{C_{s-2}}{(r-1)!(s-r-2)!}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j}\left[\bar{F}(x)\right]^{m}f(x)g_{m}^{r-1}\left[F(x)\right].$$

$$\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-2}\left[\bar{F}(y)\right]^{\gamma_{s-1}-1} f(y)\,dydx = \frac{jC_{s-1}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}}{\alpha\theta\beta\gamma_s\lambda(r-1)!(s-r-1)!v!}$$

$$\int_0^{\infty}\int_x^{\infty} x^i y^{j-\theta(1-v)}\left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1} f(y)\,dydx.$$

By integrating the first term in the left hand side by parts, the expression will be

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty} x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_s} dydx = \frac{jC_{s-1}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v}}{\alpha\theta\beta\gamma_s\lambda(r-1)!(s-r-1)!v!}\int_0^{\infty}\int_x^{\infty} x^i y^{j-\theta(1-v)}\left[\bar{F}(x)\right]^m$$

$$f(x)\, g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1} f(y)\,dydx.$$

This is implies that

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^{\infty}\int_x^{\infty} x^i y^{j-1}\left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}\left[F(x)\right]\left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1}$$

$$\left[\bar{F}(y)\right]^{\gamma_s-1}\left\{\bar{F}(y)-\frac{1}{\alpha\theta\beta v!}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}(-1)^{u+v}(\beta+u)^{v}\lambda^{v} f(y)\right\}dydx = 0. \tag{27}$$

Now by applying a generalization of the Muntz-Szasz theorem (see, Hwang and Lin (1984)) to Eq.(27), we get

$$\bar{F}(x)-\frac{1}{\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}\lambda^{v-1}}{v!}\, y^{1-\theta(1-v)} f(y) = 0,$$

Hence,

$$\bar{F}(y) = \frac{1}{\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(-1)^{u+v}(\beta+u)^{v}\lambda^{v-1}}{v!}\, y^{1-\theta(1-v)} f(y)$$

$$= \sum_{u=0}^{\infty}\binom{1-\beta}{u}(-1)^{u}\frac{e^{-(\beta+u)\lambda y^{\theta}}}{\alpha\beta\lambda\theta y^{\theta-1}} f(x).$$

Therefore,

$$\bar{F}(y) = \frac{e^{-\lambda y^{\theta}} f(y)}{\alpha\beta\theta\lambda y^{\theta-1}}\left(e^{\lambda y^{\theta}}-1\right)^{1-\beta}.$$

Integrating both side from 0 to y, we ge

$$\int_0^{y}\frac{f(x)}{\bar{F}(x)}dx = \alpha\beta\theta\lambda\int_0^{y} x^{\theta-1}e^{\lambda x^{\theta}}\left(e^{\lambda x^{\theta}}-1\right)^{\beta-1}dx$$

This is implies that

$$-\ln\left[\bar{F}(y)\right] = \alpha\left(e^{\lambda y^{\theta}}-1\right)^{\beta}$$

$$\bar{F}(y) = \mathrm{e}^{-\alpha\left(e^{\lambda y^{\theta}}-1\right)^{\beta}}; \qquad y \ge 0.$$

Remark 4.1 By putting $m=0,\ k=1$ in Eq. (26), the recurrence relations for product moments of order statistics are obtained as

$$E\left[X_{r,s:n}^{i,j}\right]-E\left[X_{r,s-1:n}^{i,j}\right]=\frac{j}{\alpha\beta\theta(n-s+1)}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(\beta+u)^{v}}{v!}E\left[X_{r,s:n}^{i,j-\theta(1-v)}\lambda^{v-1}\right]. \tag{28}$$

Remark 4.2 By setting $m=-1$ in Eq. (26), the recurrence relations for product moments of k^{th} record values are given as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right]-E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right]$$

$$=\frac{j}{k\alpha\beta\theta}\sum_{u,v=0}^{\infty}\binom{1-\beta}{u}\frac{(\beta+u)^{v}}{v!}E\left[\left(X_r^{(k)}\right)^i\lambda^{v-1}\left(X_s^{(k)}\right)^{j-\theta(1-v)}\right]. \tag{29}$$

4. Conclusion

In this paper, we have studied the characterizations of a distribution called Weibull–Weibull distribution based on recurrence relations for single and product moments of generalized order statistics. These relations are useful to compute the moments for any value of the parameters. Also, the mean and variance of order statistics for the WWD are computed for different values of parameters.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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