Estimation for Exponentiated Generalized Xgamma Distribution
from New General Class Based on Dual Generalized Order Statistics
(A.M. Abd AL-Fattah, R.E. Abd EL-Kader, G.R. AL-Dayian & A.A. EL-Helbawy)

Abstract

In this paper, the exponentiated generalized general class of distributions is introduced. Bayes estimators for the parameters, reliability and hazard rate functions are derived based on dual generalized order statistics. As a special model from exponentiated generalized general class distributions, Bayesian estimation for the unknown parameters, reliability and hazard rate functions of exponentiated generalized xgamma distribution based on dual generalized order statistics are considered. All results are specialized to lower records, also a numerical study is given to illustrate the theoretical procedures.

Keywords: Exponentiated generalized distributions; Bayesian estimation; Dual generalized order statistics; Exponentiated generalized xgamma distribution.

1. Introduction

The statistical literature contains many classes of distributions that have been constructed by extending the common families of continuous distributions providing more flexibility as far as applications is concerned. These new families have been used for modeling data in many applied areas such as engineering, economics, biological studies, environmental sciences lifetime analysis, finance and insurance. These generalized (G) distributions give more flexibility by adding new parameters to the baseline model and are useful in obtaining general results that could be applied to special cases to obtain new results.[Alzaatreh et al. (2013) and Alizadeh et al. (2017)].

There are several ways to generate new distributions, such as adding a positive parameter to a general survival function by Marshall and Olkin (1997). Also, Gupta et al. (1998) introduced the exponentiated (E) exponential distribution; by exponentiating a cumulative distribution function with a positive real number, beta generalized-G family by Eugene et al. (2002), G gamma generated distributions by Zografos and Balakrishnan (2009). Cordeiro and de Castro (2011) presented a class of Kumaraswamy G distribution and Alzaatreh et al. (2013) introduced a new method for generating families of continuous distributions.

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Cordeiro et al. (2013) proposed a class of distributions as an extension of the E type distribution which has greater flexibility of its tails and can be widely applied in many areas of biology and engineering. Given a non-negative continuous random variable $X$, the cumulative distribution function (cdf) of the Exponentiated Generalized (EG) general class of distribution is defined by

$$F(x; \alpha, \beta) = [1 - (1 - G(x))^\alpha]^\beta, \ \alpha, \beta > 0,$$  \hspace{1cm} (1)

where $\alpha$ and $\beta$ are additional shape parameters, the corresponding probability density function (pdf) for (1) is given by

$$f(x; \alpha, \beta) = \alpha \beta g(x) (1 - G(x))^{\alpha-1} [1 - (1 - G(x))^\alpha]^{\beta-1}, \ \alpha, \beta > 0.$$  \hspace{1cm} (2)

By setting $\alpha = 1$ in (1) the E type distributions is derived by Gupta et al. (1998); further the E exponential and E gamma distributions can be obtained if $G(x)$ is the exponential or gamma cumulative distributions, respectively. For $\beta = 1$ in (1) and if $G(x)$ is the Gumbel or Fréchet cumulative distributions, then, one can get the E Gumbel and E Fréchet distributions, respectively, as defined by Nadarajah and Kotz (2006). Thus, the class of distributions (1) extends both E type distributions.

General class of distributions is important to obtain a general result that could be applied to special cases in obtaining new results. It is more flexible in dealing with statistical problems. Many authors focused on the G and EG distributions and its applications; for example, Oguntunde et al. (2014), Yousof et al. (2015), De Andrade et al. (2016), Mustafa et al. (2016) and Sindi et al. (2017).

AL-Hussaini (1999) introduced a general class of distributions with positive domain to be the underlying population model. This class of distributions includes, the Weibull, Pareto Type I, beta Type I, Gompertz and Compound Gompertz distribution, among others.

$$G_1(x|\theta) = 1 - \exp[-\lambda(x)], \ \ \ \ x > 0, \hspace{1cm} (3)$$

where $\lambda(x) \equiv \lambda(x, \theta)$ is a non-negative continuous function of $x$ such that $\lambda(x, \theta) \to 0$ as $x \to 0^+$ and $\lambda(x, \theta) \to \infty$ as $x \to \infty$, and $\theta = (\theta_1, \theta_2, ..., \theta_k)$.

In this paper the new exponentiated generalized general class (EGGC) of distributions will be introduced; using (1) and (3), the cdf and pdf can be derived as follows:

$$F(x; \alpha, \beta) = [1 - \exp[-\alpha \lambda(x)]]^\theta, \ x > 0, \ \alpha, \beta > 0,$$ \hspace{1cm} (4)

and
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\[ f(x; \alpha, \beta) = ab \lambda(x) \exp[-\alpha \lambda(x)](1 - \exp[-\alpha \lambda(x)])^{\beta-1}, x > 0, \alpha, \beta > 0, \]  

where \( \lambda(x) \) is the derivative of \( \lambda(x) \) with respect to \( x \).

The reliability function (rf), hazard rate function (hrf) and reversed hazard rate function (rhrf) are given, respectively, by

\[ R(x) = 1 - [1 - \exp[-\alpha \lambda(x)]]^\beta, \alpha, \beta, x > 0, \]  

\[ h_1(x) = \frac{f(x)}{R(x)} = \frac{a \beta \lambda(x) \exp[-\alpha \lambda(x)](1 - \exp[-\alpha \lambda(x)])^{\beta-1}}{1 - [1 - \exp[-\alpha \lambda(x)]]^\beta}, x > 0; \alpha, \beta > 0, \]  

and

\[ h_2(x) = \frac{f(x)}{F(x)} = \alpha \beta \lambda(x) \exp[-\alpha \lambda(x)](1 - \exp[-\alpha \lambda(x)])^{-1}, x > 0; \alpha, \beta > 0. \]

Burkschat et al. (2003) studied the dual generalized order statistics (dgos) that enables a common approach to descending ordered random variables as reversed ordered order statistics, lower record models and lower Pfeifer records.

Let \( X_{(1,n,m,k)}, X_{(2,n,m,k)}, \ldots, X_{(n,n,m,k)} \) be \( n \) dgos from an absolutely cdf with corresponding pdf. Then, the joint pdf has the form

\[ f_{X_{(1,n,m,k)}, X_{(2,n,m,k)}, \ldots, X_{(n,n,m,k)}}(x_{(1)}, \ldots, x_{(n)}) = \]

\[ k \left( \prod_{j=1}^{k-1} y_j \right) \left( \prod_{l=1}^{\gamma} \left( F(x_{(l)}) \right) \right)^m f(x_{(1)}) \left( f(x_{(n)}) \right)^{-1}, \]  

where \( F^{-1}(1) \geq x_{(1)} \geq \cdots \geq x_{(n)} \geq F^{-1}(0), n \in N, k \geq 1, m_1, \ldots, m_{n-1} = m, \)

\( m \in R \) is the parameters such that \( y_r = k + (n-r)(m+1) \geq 1, \) for all \( 1 \leq r \leq n. \)

This paper is organized as follows: in Section 2, Bayes estimators for the parameters, rf and hrf of EGGC of distributions based on dgos under squared error (SE) and linear exponential (LINEX) loss functions are derived. The exponentiated generalized xgamma (EG-Xg) distribution is studied in details as an application for this general class in Section 3. Finally a numerical study is presented in Section 4, to illustrate the application procedures of the various results developed in this paper.
2. Bayesian Estimation Based on Dual Generalized Order Statistics

This section is devoted to estimate the parameters, rf and hrf based on dgos using Bayesian approach, under SE and LINEX loss functions. Also the credible intervals are obtained.

Suppose that \( X_{(1,n,m,k)}, X_{(2,n,m,k)}, \ldots, X_{(n,n,m,k)} \) are \( n \) dgos from EGGC distribution, the likelihood function can be obtained by substituting (4) and (5) in (9) as follows:

\[
L(\alpha, \beta; \mathbf{x}) = k \left( \prod_{i=1}^{n-1} \gamma_i \right) \alpha^n \beta^n \prod_{i=1}^{n} \lambda(x_i) \exp \left[ -\alpha \sum_{i=1}^{n} \lambda(x_i) \right] \prod_{i=1}^{n-1} \left[ 1 - \exp[-\alpha \lambda(x_i)] \right]^{\beta(m+1)-1} \times \left[ 1 - \exp[-\alpha \lambda(x_n)] \right]^{\beta k-1}.
\]  

(10)

Assuming that the parameters \( \alpha \) and \( \beta \) of EGGC distributions are random variables with a joint bivariate prior density function that was used by AL-Hussaini and Jaheen (1992) as

\[
g(\alpha, \beta) = g_1(\beta|\alpha)g_2(\alpha), \tag{11}
\]

where \( g_1(\beta|\alpha) = \frac{\alpha^{\tau+1} e^{-\frac{\alpha}{w} \beta \tau}}{\Gamma(\tau+1)w^{\tau+1}} \), \( \tau > 0 \), \( w, \alpha, \beta > 0 \),  

(12)

and the prior of \( \alpha \) is

\[
g_2(\alpha) = \frac{\alpha^{c-1} e^{-\frac{\alpha}{b} \gamma}}{\Gamma(c) b^c} \), \quad \alpha, b, c > 0, \tag{13}
\]

which is the gamma \((c, b)\) density.

The joint prior pdf of \( \alpha \) and \( \beta \), will be obtained by substituting (12) and (13) in (11) and it's given by;

\[
g(\alpha, \beta) \propto \alpha^{c+\tau} \beta^{\tau} e^{-\alpha(\frac{1}{b} \beta \tau + \frac{1}{w})}, \quad c, b, w > 0, \tau > -1. \tag{14}
\]

The joint posterior of \( \alpha \) and \( \beta \) can be derived by using (10) and (14) as follows:

\[
\pi(\alpha, \beta|\mathbf{x}) \propto L(\alpha, \beta|\mathbf{x})g(\alpha, \beta)
\]

\[
\pi(\alpha, \beta|\mathbf{x}) \propto \alpha^{n+\tau+c} \beta^{n+\tau} e^{-\alpha(\frac{1}{b} \beta \tau + \frac{1}{w}) \lambda(x_n)} e^{-\beta (m+1) \sum_{i=1}^{n} \lambda(x_i) \ln u_i - (m+1) \sum_{i=1}^{n} \ln u_i \ln u_n} \prod_{i=1}^{n} (u_i)^{-1},
\]

(15)

where \( u_i = [1 - \exp(-\alpha \lambda(x_i))] \) and \( u_n = [1 - \exp(-\alpha \lambda(x_n))] \).

Hence, the joint posterior distribution of \( \alpha \) and \( \beta \) is given by;

\[
\pi(\alpha, \beta|\mathbf{x}) = \frac{\alpha^{n+\tau+c} \beta^{n+\tau} e^{-\alpha(\frac{1}{b} \beta \tau + \frac{1}{w}) \lambda(x_n)} e^{-\beta (m+1) \sum_{i=1}^{n} \lambda(x_i) \ln u_i - (m+1) \sum_{i=1}^{n} \ln u_i \ln u_n} \prod_{i=1}^{n} (u_i)^{-1}}{\Gamma(n+\tau+1) \varphi},
\]

(16)
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\[ \varphi = \int_{0}^{\infty} a^{n+r+c} e^{-\frac{a}{w} x^{n+1}} \frac{\prod_{i=1}^{n} u_{i}^{-1}}{w^{n+1} \ln u_{i} - k \ln u_{n}} d\alpha. \]  

(17)

2.1 Point estimation

In this subsection the Bayes estimators for the parameters, rf and hrf based on dgos under SE and LINEX loss functions of the EGGC distribution are obtained.

a. Bayesian estimation of the parameters, rf and hrf under squared error loss function

Under SE loss function the Bayes estimators of the parameters \( \alpha \) and \( \beta \) are given by their marginal posterior expectations using (16) as shown below

\[ \alpha_{SE}^{*} = E(\alpha|x) \]

\[ = \int_{0}^{\infty} \frac{1}{\varphi \left( \frac{a}{w} - (m+1) \sum_{i=1}^{n} \ln u_{i} - k \ln u_{n} \right)} a^{n+r+c+1} e^{-\alpha \left( \frac{a}{w} + \sum_{i=1}^{n} \lambda_{i} \right)} \prod_{i=1}^{n} \left( u_{i} \right)^{-1} d\alpha, \]

(18)

and

\[ \beta_{SE}^{*} = E(\beta|x) \]

\[ = \int_{0}^{\infty} \left( \frac{n+r+1}{\varphi \left( \frac{a}{w} - (m+1) \sum_{i=1}^{n} \ln u_{i} - k \ln u_{n} \right)} \right) \frac{\prod_{i=1}^{n} \left( u_{i} \right)^{-1}}{1} d\alpha. \]

(19)

The Bayes estimators of the rf and hrf under SE loss function are as follows:

\[ R_{SE}^{*}(x) = E(R(x)|x) \]

\[ = 1 - \int_{0}^{\infty} \frac{a^{n+r+c} e^{-\alpha \left( \frac{a}{w} + \sum_{i=1}^{n} \lambda_{i} \right)} \prod_{i=1}^{n} \left( u_{i} \right)^{-1}}{\varphi \left( \frac{a}{w} - (m+1) \sum_{i=1}^{n} \ln u_{i} - k \ln u_{n} \right)^{n+r+1}} d\alpha, \]

(20)

and

\[ h_{1}^{*}(SE)(x) = E(h(x)|x) \]

\[ = \int_{0}^{\infty} \frac{a^{n+r+c+1} \beta^{n+r+1} \lambda(x) \exp \left( -\alpha \lambda(x) \right) \prod_{i=1}^{n} \left( u_{i} \right)^{-1}}{(1-u^{\beta})(n+r+1) \varphi} \left( 1-u^{\beta} \right)^{n+r+1} \prod_{i=1}^{n} \left( u_{i} \right)^{-1} d\beta. \]

(21)

where \( u = \left[ 1 - \exp \left( -\alpha \lambda(x) \right) \right] \).

(22)

and \( u_{i} \) and \( u_{n} \) are given by (15).
To obtain the Bayes estimates of the parameters, rf and hrf, the Equations (18) - (21) should be solved numerically.

**b. Bayesian estimation of the parameters, rf and hrf under linear exponential loss function**

Under the LINEX loss function, the Bayes estimators for the shape parameters $\alpha$ and $\beta$ are given, respectively, by

$$
\alpha^{*}_{(\text{LINEX})} = -\frac{1}{\theta} \ln E\left(e^{-\theta \alpha}|x\right),
$$  
(23)

where

$$
E\left(e^{-\theta \alpha}|x\right) = \int_{0}^{\infty} \frac{\alpha^{n+\tau+c} e^{-\alpha (\theta + \frac{1}{\theta} \sum_{i=1}^{n} \lambda(x_i))} \prod_{i=1}^{n} (u_i)^{-1}}{\varphi \left( \frac{\alpha}{\theta} - (m+1) \sum_{i=1}^{n-1} \ln u_i - k \ln u_n \right)^{n+\tau+1}} \, d\alpha,
$$  
(24)

and

$$
\beta^{*}_{(\text{LINEX})} = -\frac{1}{\theta} \ln E\left(e^{-\theta \beta}|x\right),
$$  
(25)

where

$$
E\left(e^{-\theta \beta}|x\right) = \int_{0}^{\infty} \frac{\alpha^{n+\tau+c} \beta^{n+\tau+\theta} (1-u)^\theta e^{-\alpha (\theta + \frac{1}{\theta} \sum_{i=1}^{n} \lambda(x_i))} e^{-\beta (\theta + \frac{1}{\theta} \sum_{i=1}^{n-1} \ln u_i - k \ln u_n)}}{\Gamma(n+\tau+1) \varphi \left( \frac{\alpha}{\theta} - (m+1) \sum_{i=1}^{n-1} \ln u_i - k \ln u_n \right)^{n+\tau+1}} \, d\alpha \, d\beta,
$$  
(26)

Also, the Bayes estimator for rf based on dgos can be obtained as follows:

$$
R^{*}_{(\text{LINEX})}(x) = -\frac{1}{\theta} \ln E\left(e^{-\theta R(x)}|x\right),
$$  
(27)

where

$$
E\left(e^{-\theta R(x)}|x\right) =
1 - \int_{0}^{\infty} \int_{0}^{\infty} \frac{\alpha^{n+\tau+c} \beta^{n+\tau+\theta} (1-u)^\theta e^{-\alpha (\theta + \frac{1}{\theta} \sum_{i=1}^{n} \lambda(x_i))} e^{-\beta (\theta + \frac{1}{\theta} \sum_{i=1}^{n-1} \ln u_i - k \ln u_n) \prod_{i=1}^{n} (u_i)^{-1}}}{\Gamma(n+\tau+1) \varphi \left( \frac{\alpha}{\theta} - (m+1) \sum_{i=1}^{n-1} \ln u_i - k \ln u_n \right)^{n+\tau+1}} \, d\alpha \, d\beta,
$$  
(28)

and the Bayes estimators for hrf and rhrf based on dgos can be derived as given below:

$$
h^{*}_{1(\text{LINEX})}(x) = -\frac{1}{\theta} \ln E\left(e^{-\theta h_1(x)}|x\right),
$$  
(29)

where
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\[ E(e^{-\theta h_1(x)}|x) = \int_0^\infty \int_0^\infty e^{ax+c\beta+x\beta} \left[ \frac{\lambda(x)e^{-(ax+c\beta)x}x^a}{\Gamma(a+\beta+1)} \right] e^{-\beta\gamma - (m+1)\sum_{i=1}^{m} \ln u_i - k\ln u_0} \prod_{i=1}^{m} (u_i)^{-1} \, da \, d\beta, \]  

(30)

and

\[ h_2^*(LNX)(x) = \frac{-1}{\theta} \ln E(e^{-\theta h_1(x)}|x). \]  

(31)

where

\[ E(e^{-\theta h_2(x)}|x) = \int_0^\infty \int_0^\infty e^{ax+c\beta+x\beta} e^{-\theta \lambda(x)e^{-(ax+c\beta)x}x^a} \left[ \frac{\lambda(x)e^{-(ax+c\beta)x}x^a}{\Gamma(a+\beta+1)} \right] e^{-\beta\gamma - (m+1)\sum_{i=1}^{m} \ln u_i - k\ln u_0} \prod_{i=1}^{m} (u_i)^{-1} \, da \, d\beta. \]  

(32)

To obtain the Bayes estimators of the parameters, rf and hrf, the Equations (23) - (32) should be solved numerically.

2.2 Credible intervals

In this subsection the Bayesian analog to the confidence interval which is called a credible intervals are introduced. In general, \((L(x), U(x))\) is \(100(1 - \omega)\%\) credible interval of \(\theta\) if

\[ P[L(x) < \theta < U(x)|x] = \int_{L(x)}^{U(x)} \pi^*(\theta|x) \, d\theta = 1 - \omega. \]  

(33)

Since, the posterior distribution is given by (16), then a \(100 (1 - \omega) \%\) credible interval for \(\alpha\) is \((L(x), U(x))\), where

\[ P[\alpha > L(x)|x] = \int_{L(x)}^{\infty} \frac{\alpha^{a+c\beta} e^{-\alpha^{a+c\beta}} \prod_{i=1}^{m} (u_i)^{-1} \prod_{i=1}^{m} (u_i)^{-1} \, da}{\Gamma(a+\beta+1)} = 1 - \frac{\omega}{2}, \]  

(34)

and

\[ P[\alpha > U(x)|x] = \int_{u(U(x))}^{\infty} \frac{\alpha^{a+c\beta} e^{-\alpha^{a+c\beta}} \prod_{i=1}^{m} (u_i)^{-1} \prod_{i=1}^{m} (u_i)^{-1} \, da}{\Gamma(a+\beta+1)} = \frac{\omega}{2}. \]  

(35)

Also, \(100 (1 - \omega) \%\) credible interval for \(\beta\) is \((L(x), U(x))\), where

\[ P[\beta > L(x)|x] = \int_{L(x)}^{\infty} \int_0^\infty \frac{e^{a+c\beta+c\beta} e^{-\alpha^{a+c\beta}} \prod_{i=1}^{m} (u_i)^{-1} \prod_{i=1}^{m} (u_i)^{-1} \, da \, d\beta}{\varphi \Gamma(n + \tau + 1)} = 1 - \frac{\omega}{2}, \]  

(36)

and

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\[ P[\beta > U(x)|x] = \int_0^\infty \frac{x^{n+\tau+\beta - 1} e^{-\frac{x^2}{2}} \Gamma(n+1) \Gamma(n+1) \prod_{k=1}^n \Gamma(n_k)}{\phi(n+\tau+1)} \prod_{k=1}^n \Gamma(n_k)^{-1} \, da \, db \]
\[ = \frac{\omega}{2}. \]  
\[ (37) \]

3. Exponentiated Generalized Xgamma Distribution Based on Dual Generalized Order Statistics

Sen et al. (2016) introduced the xgamma (xg) distribution which is generated as a special finite mixture of exponential(\(\theta\)) and gamma(3, \(\theta\)) distributions with mixing proportion \(\pi_1 = \frac{\theta}{1+\theta}\) and \(\pi_2 = 1 - \pi_1 = \frac{1}{1+\theta}\). The cdf and pdf of the xgamma distribution are, respectively

\[ F_{XG}(x; \theta) = 1 - \frac{1+\theta + \theta x + \theta^2 x^2}{2} e^{-\theta x}, \quad x > 0, \quad \theta > 0, \]  
\[ (38) \]

and

\[ f_{XG}(x; \theta) = \frac{\theta^2 x}{1+\theta} \left( 1 + \frac{\theta x}{2} e^{-\theta x}, \quad x > 0, \quad \theta > 0. \]  
\[ (39) \]

Yadav et al. (2018) studied the generalized xgamma (G-xg) distribution by adding power shape parameter to the cdf; some statistical properties of this G-xg are discussed. They used many methods of estimation to estimate the r.f and hrf of the G-xg distribution.

Assuming that \(X\) is a random variable distribution with EG-xg distribution with shape parameters, \(\alpha, \beta > 0\) and scale parameter \(\theta > 0\) denoted by \(X \sim \text{EG}xg(\alpha, \beta, \theta)\), hence the pdf and cdf are given, respectively, by

\[ F_{EGxg}(x; \alpha, \beta, \theta) = \left[ 1 - \frac{(1+\theta + \theta x + \theta^2 x^2)\alpha}{(1+\theta)^\alpha} e^{-\alpha \theta x} \right]^\beta, \quad x > 0, \quad \alpha, \beta, \theta > 0. \]  
\[ (40) \]

and

\[ f_{EGxg}(x; \alpha, \beta, \theta) = \frac{\alpha \beta \theta^2 x^{\alpha-1} e^{-\alpha \theta x}}{(1+\theta)^\alpha} \left( 1 + \frac{\theta x}{2} \right) \left( 1 + \theta + \theta x + \frac{\theta^2 x^2}{2} \right)^{\alpha-1} \]
\[ \times \left[ 1 - \frac{(1+\theta + \theta x + \theta^2 x^2)\alpha}{(1+\theta)^\alpha} e^{-\alpha \theta x} \right]^{\beta-1}, \quad x > 0, \quad \alpha, \beta, \theta > 0. \]  
\[ (41) \]

The rf, hrf and rhrf are, respectively, given by
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\[ R_{EGxg}(x; \alpha, \beta, \theta) = 1 - \left[ 1 - \frac{(1 + \theta + \theta x + \frac{\theta^2 x^2}{2})^\alpha}{(1 + \theta)^2} e^{-\alpha x} \right]^\beta, \quad x > 0, \; \alpha, \beta, \theta > 0, \quad (42) \]

\[ h_{1EGxg}(x; \alpha, \beta, \theta) = \frac{\alpha \beta^2 e^{-\alpha x}}{(1 + \theta)^2} \left( 1 + \frac{\theta x^2}{2} \right) \left( 1 + \frac{\theta x + \frac{\theta^2 x^2}{2}}{2} \right)^{\alpha-1} \left[ 1 - \frac{(1 + \theta + \theta x + \frac{\theta^2 x^2}{2})^\alpha}{(1 + \theta)^2} e^{-\alpha x} \right]^{\beta-1} \]

\[ = \frac{1}{1 - \left[ 1 - \frac{(1 + \theta + \theta x + \frac{\theta^2 x^2}{2})^\alpha}{(1 + \theta)^2} e^{-\alpha x} \right]^\beta}, \quad x > 0, \; \alpha, \beta, \theta > 0, \quad (43) \]

and

\[ h_{2EGxg}(x; \alpha, \beta, \theta) = \frac{\alpha \beta^2 e^{-\alpha x}}{(1 + \theta)^2} \left( 1 + \frac{\theta x^2}{2} \right) \left( 1 + \frac{\theta x + \frac{\theta^2 x^2}{2}}{2} \right)^{\alpha-1} \times \left[ 1 - \frac{(1 + \theta + \theta x + \frac{\theta^2 x^2}{2})^\alpha}{(1 + \theta)^2} e^{-\alpha x} \right]^{-1}, \quad x, \alpha, \beta, \theta > 0. \quad (44) \]

Plots of the pdf, hrf and rhrf of EG-xg are, respectively, given in Figures 1-3. The plots, in Figure 1 indicate the behavior of the density function and explain the flexibility of the model graphically with its sub-families. From Figure 1, one can observe that the curves of the pdf are monotone decreasing, increasing and bathtub, with different values of shape and scale parameters. In Figure 2, the curves of the hrf are increasing, decreasing and monotone decreasing with different values of shape and scale parameters. The curves of the rhrf at the all values are decreasing and then constant in Figure 3.
Figure 1
Plots of the probability density function

Figure 2
Plots of the hazard rate function

Figure 3
Plots of the reversed hazard rate function
3.1 Bayesian estimation for exponentiated generalized xgamma distribution

This section derived the estimation of the parameters, rf and hrf based on dgos using Bayesian approach. Also the credible intervals of the parameters, rf and hrf are obtained.

Suppose that \( X_{(1,n,m,k)}, X_{(2,n,m,k)}, \ldots, X_{(n,n,m,k)} \) are \( n \) dgos from EG-Xg distribution, the likelihood function can be derived by substituting (40) and (41) in (9) as follows:

\[
L_{EGXg}(\alpha, \beta, \theta; x) = k \left( \prod_{i=1}^{n} \gamma_{i} \right) \alpha^{n} \beta^{n} \theta^{2n} e^{-\alpha \theta \sum_{i=1}^{n} x_{i}} \prod_{i=1}^{n} \left( 1 + \frac{\theta x_{i}^{2}}{2} \right) \delta_{i}^{x_{i} - 1} \\
\times \prod_{i=1}^{n} \left[ 1 - \left( \frac{\delta_{i}}{(1 + \theta)} \right)^{\alpha} e^{-\alpha \theta x_{i}} \right]^{\beta(m+1)-1} \left[ 1 - \left( \frac{\delta_{n}}{(1 + \theta)} \right)^{\alpha} e^{-\alpha \theta x_{n}} \right]^{\beta k-1},
\]

(45)

where \( \delta_{i} = \left( 1 + \theta + \theta x_{i} + \frac{\theta^{2} x_{i}^{2}}{2} \right) \) and \( \delta_{n} = \left( 1 + \theta + \theta x_{n} + \frac{\theta^{2} x_{n}^{2}}{2} \right) \).

(46)

3.1.1 Point estimation

In this subsection, the Bayesian approach is considered under SE and LINEX loss functions to estimate the parameters, rf and hrf of the EG-xg distribution based on dgos. Also credible intervals for the parameters are obtained.

Let \( \theta_{1} = \alpha, \theta_{2} = \beta \) and \( \theta_{3} = \theta \) are independent random variables with gamma prior distribution with the pdf as follows:

\[
\pi(\theta) = \frac{d_{j}^{c_{j}}}{\Gamma(c_{j})} \theta_{j}^{c_{j}-1} e^{-d_{j} \theta_{j}}, \quad \theta_{j}, d_{j}, c_{j} > 0, \ j = 1, 2, 3, \text{where } c_{j}, d_{j} \text{ are the hyper parameters.}
\]

A joint prior density function of \( \theta = (\theta_{1}, \theta_{2}, \theta_{3})' \) is then given by

\[
\pi(\theta) \propto \prod_{j=1}^{3} \theta_{j}^{c_{j}-1} e^{-d_{j} \theta_{j}}.
\]

(47)

The likelihood function given by (45) can be rewritten as

\[
L_{EGXg}(\alpha, \beta, \theta; x) = k \left( \prod_{j=1}^{n} \gamma_{j} \right) \alpha^{n} \beta^{n} \theta^{2n} e^{-\alpha \theta \sum_{i=1}^{n} x_{i}} \prod_{i=1}^{n} \left( 1 + \frac{\theta x_{i}^{2}}{2} \right) \delta_{i}^{x_{i} - 1} \\
\times \prod_{i=1}^{n} \left[ 1 - \left( \frac{\delta_{i}}{(1 + \theta)} \right)^{\alpha} e^{-\alpha \theta x_{i}} \right]^{\beta(m+1)-1} \left[ 1 - \left( \frac{\delta_{n}}{(1 + \theta)} \right)^{\alpha} e^{-\alpha \theta x_{n}} \right]^{\beta k-1},
\]

(48)

where \( u_{i} = \left[ 1 - \left( \frac{\delta_{i}}{(1 + \theta)} \right)^{\alpha} e^{-\alpha \theta x_{i}} \right], u_{n} = \left[ 1 - \left( \frac{\delta_{n}}{(1 + \theta)} \right)^{\alpha} e^{-\alpha \theta x_{n}} \right] \) and \( \rho_{i} = \left( 1 + \frac{\theta x_{i}^{2}}{2} \right) \delta_{i}^{x_{i} - 1} \).

The joint posterior density can be derived by using (47) and (48) as follows:
\[ \pi_{EGxg}(\theta|x) \\
= T(a^{c_1+n-1}\beta c_2+n-1\theta c_3+2n-1)e^{-\alpha(d_1+\theta \sum_{i=1}^{n} x_i+\text{ln}(1+\theta))} \\
\times e^{-\beta [d_2-(m+1)\sum_{i=1}^{n} \text{ln}(u_i)-\text{ln}(u_n)]} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^{-1}), \] (49)

where \( T \) is the normalizing constant defined by

\[ T^{-1} = \int_0^{\infty} \int_0^{\infty} \left( a^{c_1+n-1}\beta c_2+n-1\theta c_3+2n-1 e^{-\alpha(d_1+\theta \sum_{i=1}^{n} x_i+\text{ln}(1+\theta))} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^{-1}) \right) \\
\times \left[ \int_0^{\infty} e^{-\beta [d_2-(m+1)\sum_{i=1}^{n} \text{ln}(u_i)-\text{ln}(u_n)]} \, d\theta \right] \, da \, d\beta. \]

\[ = \Gamma(c_2+n) \int_0^{\infty} \int_0^{\infty} \left( a^{c_1+n-1}\beta c_2+n-1\theta c_3+2n-1 e^{-\alpha(d_1+\theta \sum_{i=1}^{n} x_i+\text{ln}(1+\theta))} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^{-1}) \right) \\
\times \left[ \int_0^{\infty} e^{-\beta [d_2-(m+1)\sum_{i=1}^{n} \text{ln}(u_i)-\text{ln}(u_n)]} \, d\theta \right]. \]

Let

\[ \psi = \int_0^{\infty} \int_0^{\infty} \left( a^{c_1+n-1}\beta c_2+n-1\theta c_3+2n-1 e^{-\alpha(d_1+\theta \sum_{i=1}^{n} x_i+\text{ln}(1+\theta))} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^{-1}) \right) \\
\times \left[ \int_0^{\infty} e^{-\beta [d_2-(m+1)\sum_{i=1}^{n} \text{ln}(u_i)-\text{ln}(u_n)]} \, d\theta \right] \, da. \]

Hence, the joint posterior distribution of \( \alpha, \beta \) and \( \theta \) given \( x \) can be written as;

\[ \pi_{EGxg}(\alpha, \beta, \theta | x) = \\
\left( a^{c_1+n-1}\beta c_2+n-1\theta c_3+2n-1 \right) e^{-\alpha(d_1+\theta \sum_{i=1}^{n} x_i+\text{ln}(1+\theta))} e^{-\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^{-1})^{-1} e^{-\beta [d_2-(m+1)\sum_{i=1}^{n} \text{ln}(u_i)-\text{ln}(u_n)]} \\
\phi \psi \Gamma(c_2+n), \] (50)

**a. Bayesian estimation of the parameters, rf and hrf under squared error loss function**

Under SE loss function the Bayes estimators of the parameters \( \alpha, \beta \) and \( \theta \) are given by their marginal posterior expectations using (50) as shown below

\[ \alpha_{(SE)EGxg} = E(\alpha | x) = \int_0^{\infty} \int_0^{\infty} \frac{1}{\psi \Gamma(c_2+n)} \left( a^{c_1+n-1}\beta c_2+n-1\theta c_3+2n-1 \right) e^{-\alpha(d_1+\theta \sum_{i=1}^{n} x_i+\text{ln}(1+\theta))} \\
\times e^{-\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^{-1})^{-1} e^{-\beta [d_2-(m+1)\sum_{i=1}^{n} \text{ln}(u_i)-\text{ln}(u_n)]} \, d\beta \, d\theta \, d\alpha, \]
Estimation for Exponentiated Generalized Xgamma Distribution from New General Class Based on Dual Generalized Order Statistics
(A.M. Abd-AL-Fattah, R.E. Abd EL-Kader, G.R. AL-Dayan & A.A. El-Helbawy)

\[
\begin{align*}
\beta^*_{SEEXG} &= E(\beta|x) \\
&= \int_0^\infty \int_0^\infty \frac{1}{\varphi^*(c_2 + n)} \left( \alpha c_1 + n - 1 \beta c_2 + n \right) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1 + \theta))} e^{-\theta d_3} \prod_{i=1}^n \rho_i(u_i^*)^{-1} \\
&\times e^{-\beta(d_2 - (m + 1) \sum_{i=1}^n \ln(u_i^*) - k \ln(u_n^*))} d\beta d\alpha,
\end{align*}
\]

(51)

and

\[
\theta^*_{SEEXG} = E(\theta|x)
\]

\[
\begin{align*}
&= \int_0^\infty \int_0^\infty \frac{1}{\varphi^*(c_2 + n)} \left( \alpha c_1 + n - 1 \beta c_2 + n \right) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1 + \theta))} e^{-\theta d_3} \prod_{i=1}^n \rho_i(u_i^*)^{-1} \\
&\times e^{-\beta(d_2 - (m + 1) \sum_{i=1}^n \ln(u_i^*) - k \ln(u_n^*))} d\beta d\alpha,
\end{align*}
\]

(52)

The Bayes estimators of the rf, hrf and rhfr under SE loss function can be obtained using (42), (43) and (50) as follows:

\[
R_{(SEEXG)}(x) = E(R_{SEEXG}(x)|x) = 1 - \int_\theta (u^*)^\theta \pi^*_SEEXG(\theta|x) d\theta
\]

\[
= 1 - \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{\varphi^*(c_2 + n)} \left( \alpha c_1 + n - 1 \beta c_2 + n \right) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1 + \theta))} e^{-\theta d_3} \prod_{i=1}^n \rho_i(u_i^*)^{-1} \\
&\times e^{-\beta(d_2 - (m + 1) \sum_{i=1}^n \ln(u_i^*) - k \ln(u_n^*)}) d\beta d\alpha,
\]

(53)
Estimation for Exponentiated Generalized Xgamma Distribution  
from New General Class Based on Dual Generalized Order Statistics  
(A.M.Abd AL-Fattah, R.E.Abd EL-Kader, G.R.AL-Dayian & A.A.El-Helbawy)  

where \( u^* = \left[ 1 - \left( \frac{\sigma}{(1+\theta)} \right)^{e^{-\theta u}} \right] \)  

and  

\[
\hat{h}_{1(SE)EGxg}(x) = E(h_{1\, EGxg}(x) | x) \\
= \int \hat{h}_{1\, EGxg}(x) \pi_{EGxg}(\theta | x) \, d\theta ,
\]  

To obtain the Bayes estimators of the parameters, rf, hrf and rhf, the Equations (51) - (55) should be solved numerically.  

b. Bayesian estimation of the parameters, rf and hrf under linear exponential loss function  

Under the LINEX loss function, the Bayes estimators for the shape parameters \( \alpha, \beta \) and \( \theta \) are given, respectively, by  

\[
\hat{\alpha}_{(LNX)EGxg} = -\frac{1}{\nu} \ln E(e^{-\nu \alpha | x}) ,
\]  

where  

\[
E(e^{-\nu \alpha | x}) \\
= \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{\varphi^* \Gamma(c_2 + n)} (\alpha^{c_1+n-1} \beta c_2 + n - 1) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1+\theta) + v)} e^{-\theta d_3} \\
\times \prod_{i=1}^n \rho_i(u_i^*)^{-1} e^{-\beta [d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]} \, d\beta d\theta d\alpha ,
\]

\[
= \int_0^\infty \int_0^\infty \frac{(\alpha^{c_1+n-1} \beta c_2 + n - 1) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1+\theta) + v)} e^{-\theta d_3}}{\varphi^* [d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]} \prod_{i=1}^n \rho_i(u_i^*)^{-1} \, d\theta d\alpha ,
\]

and  

\[
\hat{\beta}_{(LNX)EGxg} = -\frac{1}{\nu} \ln E(e^{-\nu \beta | x}) ,
\]  

where  

\[
\int_0^\infty \int_0^\infty \frac{1}{\varphi^* \Gamma(c_2 + n)} (\alpha^{c_1+n-1} \beta c_2 + n - 1) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1+\theta) + v)} e^{-\theta d_3} \\
\times \prod_{i=1}^n \rho_i(u_i^*)^{-1} e^{-\beta [d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]} \, d\beta d\theta d\alpha ,
\]

\[
= \int_0^\infty \int_0^\infty \frac{(\alpha^{c_1+n-1} \beta c_2 + n - 1) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1+\theta) + v)} e^{-\theta d_3}}{\varphi^* [d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]} \prod_{i=1}^n \rho_i(u_i^*)^{-1} \, d\theta d\alpha ,
\]

\[
\int_0^\infty \int_0^\infty \frac{1}{\varphi^* \Gamma(c_2 + n)} (\alpha^{c_1+n-1} \beta c_2 + n - 1) e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(1+\theta) + v)} e^{-\theta d_3} \\
\times \prod_{i=1}^n \rho_i(u_i^*)^{-1} e^{-\beta [d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]} \, d\beta d\theta d\alpha ,
\]
\[ E(e^{-\theta \|x\|}) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{(c_1 + n - 1)c_2 + n - 1}{\varphi \Gamma(c_2 + n)} e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(i + \theta))} e^{-\theta d_3} \prod_{i=1}^n \rho_i \left(u_i^* \right)^{-1} \times e^{-\beta \left[ d_2 - (m + 1) \sum_{i=1}^n \ln(u_i^*) - k \ln(u_i^*) \right]} \, d\beta d\alpha d\theta, \]

Also
\[ \theta^{(LNX)_{EGXG}} = \frac{-1}{v} \ln E(e^{-\theta \|x\|}), \quad (58) \]

where
\[ E(e^{-\theta \|x\|}) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{(c_1 + n - 1)c_2 + n - 1}{\varphi \Gamma(c_2 + n)} e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(i + \theta))} e^{-\theta d_3} \prod_{i=1}^n \rho_i \left(u_i^* \right)^{-1} \times e^{-\beta \left[ d_2 - (m + 1) \sum_{i=1}^n \ln(u_i^*) - k \ln(u_i^*) \right]} \, d\beta d\alpha d\theta, \]

Also, the Bayes estimator for rf based on dgos can be obtained as follows:
\[ R^{(LNX)_{EGXG}}(x) = \frac{-1}{v} \ln E(e^{-\theta \|x\|}) \left| x \right\rangle, \quad (59) \]

where
\[ E(e^{-\theta \|x\|}) = \]
\[ = \int_0^\infty \int_0^\infty \int_0^\infty \frac{(c_1 + n - 1)c_2 + n - 1}{\varphi \Gamma(c_2 + n)} e^{-\alpha(d_1 + \theta \sum_{i=1}^n x_i + n \ln(i + \theta))} e^{-\theta d_3} \prod_{i=1}^n \rho_i \left(u_i^* \right)^{-1} \times e^{-\beta \left[ d_2 - (m + 1) \sum_{i=1}^n \ln(u_i^*) - k \ln(u_i^*) - ku \right]} \times \]

and the Bayes estimator for hrf and hrhf based on dgos can be derived as follows:
\[ h^{(LNX)_{EGXG}}(x) = \frac{-1}{v} \ln E(e^{-h \|x\|}) \left| x \right\rangle. \quad (60) \]

where
\[ E(e^{-h \|x\|}) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-h \|x\|} \pi_{\theta \|x\|} ^* \pi_{\theta \|x\|} \, d\beta d\alpha \]

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(32)
3.1.2 Credible interval

Since, the posterior distribution is given by (51), then a 100 (1 - ω) % credible interval for α is \((L(\bar{x}), U(\bar{x}))\),

where

\[
P[a_{EGx} > L(\bar{x}) | \bar{x}] = \int_{L(\bar{x})}^{\infty} \int_{0}^{\alpha} \frac{(\alpha c_1 + n-1) e^{-\alpha d_1} e^{\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^*)^{-1}}{\phi^*[d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]^{\frac{n}{2} + 1}} \, d\theta \, da
\]

\[
= 1 - \frac{\omega}{2}.
\]  

(61)

and

\[
P[a_{EGx} > U(\bar{x}) | \bar{x}] = \int_{L(\bar{x})}^{\infty} \int_{0}^{\alpha} \frac{(\alpha c_1 + n-1) e^{-\alpha d_1} e^{\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^*)^{-1}}{\phi^*[d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]^{\frac{n}{2} + 1}} \, d\theta \, da
\]

\[
= \frac{\omega}{2}.
\]  

(62)

Also, 100 (1 - ω) % credible interval for β is \((L(\bar{x}), U(\bar{x}))\),

where

\[
P[\beta_{EGx} > L(\bar{x}) | \bar{x}] = \int_{L(\bar{x})}^{\infty} \int_{0}^{\beta} \frac{(\alpha c_1 + n-1) e^{-\alpha d_1} e^{\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^*)^{-1}}{\Gamma(c_2 + n) \phi^*} \, d\theta \, d\beta
\]

\[
= 1 - \frac{\omega}{2},
\]  

(63)

and

\[
P[\beta_{EGx} > U(\bar{x}) | \bar{x}] = \int_{L(\bar{x})}^{\infty} \int_{0}^{\beta} \frac{(\alpha c_1 + n-1) e^{-\alpha d_1} e^{\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^*)^{-1}}{\Gamma(c_2 + n) \phi^*} \, d\theta \, d\beta
\]

\[
= \frac{\omega}{2}.
\]  

(64)

Also, 100 (1 - ω) % credible interval for θ is \((L(\bar{x}), U(\bar{x}))\),

\[
P[\theta_{EGx} > L(\bar{x}) | \bar{x}] = \int_{L(\bar{x})}^{\infty} \int_{0}^{\theta} \frac{(\alpha c_1 + n-1) e^{-\alpha d_1} e^{\theta d_3} \prod_{i=1}^{n} \rho_i(u_i^*)^{-1}}{\phi^*[d_2 - (m + 1) \sum_{i=1}^{n} \ln(u_i^*) - k \ln(u_i^*)]^{\frac{n}{2} + 1}} \, d\theta \, d\theta
\]

\[
= 1 - \frac{\omega}{2},
\]  

(65)

and
Estimation for Exponentiated Generalized Xgamma Distribution from New General Class Based on Dual Generalized Order Statistics
(A.M. Abd AL-Fattah, R.E. Abd EL-Kader, G.R.A. AL-Dayian & A.A. EL-Helbawy)

\[
P(\theta_{EGxg} > U(x)|x) = \int_{U(x)}^{\infty} \int_{0}^{\infty} \left( \frac{(x_1 + n - 1 - \theta)(x_2 + n - 1)}{\theta^{(d_2 - k_n)} \pi \Gamma_k} \right) e^{-\theta d_2} \frac{\prod_{i=1}^{k_n} \rho_i(u_i^\theta)^{-1} d\theta}{\prod_{i=1}^{k_n} \rho_i(u_i^\theta)^{-1} d\theta} = \frac{\theta}{2}.
\]

4. Numerical Results

This section aims to illustrate the theoretical results of Bayesian estimation under SE and LINEX loss functions. Numerical results are presented for EG-xg distribution based on lower record values through a simulation study and some applications.

4.1 Simulation study

The lower record values can be obtained as a special case from dogs by setting \( m = -1, \ k = 1 \); the estimation results obtained in Sections 2 and 3 can be specialized to lower records. The Bayes estimates of \( \alpha, \beta, \theta, \) rf and hrf and their average estimates, estimated risks (ERs) are computed based on lower record values through Monte Carlo simulation study according to the following steps:

a. To generate random number from EG-xg with shape parameters \( \alpha, \beta \) and scale parameter \( \theta \), the following steps may be used.
   - Specified the values \( \alpha, \beta, \theta \) and \( n \).
   - Generate \( U_i \) from uniform \((0, 1)\) distribution \((i = 1, 2, 3, ..., n)\).
   - Generate \( V_i \) from gamma \((1, \theta)\) distribution \((i = 1, 2, 3, ..., n)\).
   - Generate \( W_i \) from gamma \((3, \theta)\) distribution \((i = 1, 2, 3, ..., n)\).
   - If \( U_i \leq \frac{\theta}{1 + \theta} \), set \( X_i = V_i \), otherwise set \( X_i = W_i \).

b. For each sample size \( n \), consider the first observation is the first lower record value \( x_1 \) denote it by \( R_1 \) and the second observation \( x_2 \) denote it by \( R_2 \); which is smaller than the maximum \((x_1 > x_2)\) record and if \( x_1 \leq x_2 \) ignore it and repeat until you get sample of record values (RV) records.

c. At the number of the surviving units, the population parameter values \( \alpha, \beta, \theta \) and the hyper parameters of the prior distribution, the Bayes estimates of the parameters, rf and hrf under SE and LINEX loss functions are computed. The computations are performed using R programming language.

d. Tables 1 and 2 present the Bayes estimates under SE and LINEX loss functions of the parameters and their ERs, relative absolute biases (RAB) and credible intervals based on lower record values for different population parameter values for \( \alpha = (0.2, 2), \beta = \)
(1.3, 3) and θ = (0.9, 0.3), based on size of records \( R_v = 5, 7, 9 \) and replications \( NR = 10000 \).

e. Table 3 displays the Bayes averages and 95% confidence intervals of the rf and hrf at \( t_0 = 0.5, 1, 1.5 \) from EG-xg distribution based on lower record values for different sample size of records \( R_v = 5, 9 \), and replications \( NR = 10000 \).

4.2 Applications

In this subsection, the application to real data set is provided to illustrate the importance of the EG-xg distribution based on lower records. Table 4 displays Bayes estimates of rf, hrf and standard deviation (sd) from EG-xg distribution for the real data based on lower records. Bayes estimates of the parameters, sds and ERs for the real data based on lower records in Table 5.

To check the validity of the fitted model, Kolmogorov-Smirnov goodness of fit test is performed for each data set. The \( p \) values are given, respectively, 0.35 and 0.204. The \( p \) values in each case indicate that the model fits the data very well.

I. The application is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used for Bhaumik et al. (2009). The data are: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

II. The second data set are service times of 63 aircraft windshield from Tahir et al. (2015). The data are: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

4.3 Concluding remarks

○ It is clear from Tables 1-3 that the ERs of the Bayes estimates of the parameters, rf and hrf performs better and the length of the credible intervals get shorter when the sample size of \( R_v \) increases.

○ One can notice that the ERs for the estimates of the parameters, rf and hrf under LINEX loss function have the less values than the corresponding ERs of the estimates under SE loss function.

○ The results obtained in this paper can be modified to obtain special results for sub-models of EG-xg distribution as follows:
Estimation for Exponentiated Generalized Xgamma Distribution from New General Class Based on Dual Generalized Order Statistics
(A.M.Abd AL-Fattah, R.E.Abd EL-Kader, G.R.AL-Dayian & A.A.EL-Elhawy)

i. The exponentiated xgamma distribution, if $\alpha = 1$.
ii. The xgamma distribution, if $\alpha = 1$ and $\beta = 1$.
iii.

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Table 1
Bayes averages of the parameters and their estimated risks, relative absolute biases and credible intervals based on lower records ($\alpha = 2, \beta = 1.3, \theta = 0.9, \nu = 0.5$ and NR = 10000 )

<table>
<thead>
<tr>
<th>NV</th>
<th>Loss functions</th>
<th>Estimators</th>
<th>Average</th>
<th>ER</th>
<th>RAB</th>
<th>LL</th>
<th>UL</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>SE</td>
<td>$\alpha^*$</td>
<td>1.9986</td>
<td>2.85e-06</td>
<td>0.0006</td>
<td>1.9968</td>
<td>2.0003</td>
<td>0.0035</td>
</tr>
<tr>
<td></td>
<td>$\beta^*$</td>
<td>1.2988</td>
<td>2.25e-06</td>
<td>0.0024</td>
<td>1.2967</td>
<td>1.3003</td>
<td>0.0035</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta^*$</td>
<td>0.9021</td>
<td>5.38e-06</td>
<td>0.0009</td>
<td>0.8999</td>
<td>0.9037</td>
<td>0.0037</td>
<td></td>
</tr>
<tr>
<td></td>
<td>LINEX</td>
<td>$\alpha^*$</td>
<td>2.0010</td>
<td>2.57e-06</td>
<td>0.0005</td>
<td>1.9994</td>
<td>2.0032</td>
<td>0.0037</td>
</tr>
<tr>
<td></td>
<td>$\beta^*$</td>
<td>1.3007</td>
<td>1.08e-06</td>
<td>0.0017</td>
<td>1.2992</td>
<td>1.3019</td>
<td>0.0027</td>
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(36)
Table 2

Bayes averages for the parameters, their estimated risks, relative absolute biases and credible intervals based on lower records ($\alpha = 0.2$, $\beta = 3$, $\theta = 0.3$, $\nu = 0.5$, NR = 10000)

<table>
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<tr>
<th>No</th>
<th>Loss</th>
<th>Estimates</th>
<th>Average</th>
<th>ER</th>
<th>RAB</th>
<th>LL</th>
<th>UL</th>
<th>Length</th>
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Table 3
Bayes averages, their estimated risks, relative absolute biases and credible intervals of the rf and hrf at $t_0 = 0.5, 1, 1.5$ from EG-xg distribution based on SE and LENIX loss functions for different sample size of records $R_v$, and repetitions NR = 10000

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<th>$R_v$</th>
<th>Loss function</th>
<th>Estimate</th>
<th>Average</th>
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<th>RAB</th>
<th>LL</th>
<th>UL</th>
<th>Length</th>
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Table 5
Bayes estimates of the parameters, \( r_f \), \( h_{rf} \) and standard error from EG-xg distribution for the real data based on lower records

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<th>Estimates</th>
<th>SE</th>
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Table 5
Bayes estimates of the parameters, standard error and estimated risks for the real data based on lower records

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<th>Estimates</th>
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