Applications of Inverse Optimization for Convex Nonlinear Problems

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Abstract

The inverse nonlinear programming problems have wide applications in economics and industrial areas. For a given certain feasible point of nonlinear convex problem with linear constraints, the inverse optimization is to determine whether this point can be made optimal by adjusting the parameters value in the problem .Zhang and Xu are introduced method to solve the inverse nonlinear convex problem . In this paper, the algorithm to solve the inverse nonlinear convex problem is introduced for a case with change in the coefficient of the objective function only and two applied examples are presented .

Keywords: *Nonlinear programming problems, inverse optimization, convex separable programming problems.*

1. Introduction

Inverse optimization was introduced (Burton, D., & Toint, P. L. , 1992) , which is a relatively new area of research. Many applications have been found in different branches such as economics, industrial and sciences. In general, the optimization problem is to find $x^* \epsilon X$ such that the objective function $p(x, c)$ is optimal at x^*. In an inverse optimization problem, the parameter value of the problem can be adjusted as little as possible, so that the given feasible solution becomes optimal. (Zhang, J., & Liu, Z. , 1996) Some inverse linear programming problem was calculated and inverse linear programming problem was investigated. (Ahuja, R. K., & Orlin, J. B. , 1998) The general form of the inverse optimization was provided for linear programming problem . (Heuberger, C. , 2004) The different methods were introduced to solve the inverse constraint and unconstrained optimization problems. (Mohamed, N. , 2006) A comparative study of the inverse optimization problem was presented for single and multiple objective problems. (Moustafa, H., , 2007) The inverse of the static traffic assignment problem was introduced with constant link costs. (Zhang, J., & Xu, C. , 2010) The inverse optimization was provided for linearly constrained convex separable programming problems and obtained the necessary and sufficient condition to make the feasible

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point becomes optimal solution of the problem. (Jiang, Y., Xiao, X., Zhang, L., & Zhang, J. , 2011) A perturbation approach was used for a type of inverse linear programming problems. (Jain, S., & Arya, N. , 2013) The inverse optimization was provided for transportation problems. Most studies focus on the inverse optimization for linear programming problems and its applications but few of these researches expand to the inverse optimization for nonlinear programming problems.

In this paper, the definition and general mathematical form of inverse optimization for nonlinear convex problem with linear constraints is presented in section 2 . In section 3, The algorithm to solve the inverse nonlinear convex problem is introduced. Two applied examples are illustrated in section 4. The conclusion is presented in section 5.

2. Definition of inverse optimization for nonlinear convex programming problem

We present the inverse optimization for nonlinear convex programming problem with linear constraints and its mathematical form which was defined by (Zhang, J., & Xu, C. ,2010).

Consider the following nonlinear convex programming problem:

$$\left.\begin{array}{c} \text{Min } z = \sum_{i=1}^{n} c_i f(x_i) \qquad , \quad c_i \neq 0 \\ \text{s.t.} \\ \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji} x_i = b_j \\ x_i \geq 0 \end{array}\right] \quad (1)$$

Where x_i is represent the decision variable $i = 1, \dots, n$, $f(x_i)$ is a continuous and Convex function of x , f(x) has second derivatives, c_i is the parameter of x_i in the objective function ,a_{ji} is the parameter of x_i in the constraint j where $i = 1, \dots, n$ and $j = 1, \dots, m$ and b_jThe constant parameters in the right side. If we consider the optimal solution of the model defined in (1) is(x^*, z^*), then the corresponding inverse programming problem to the model above is to make one of the possible solutions x^0 an optimal solution as (x^0, z^*) and that can be get through the creation of a change in the transaction c_i where the maker wants the resolution to determine the optimum values for the coefficient objective function $d_i$instead of c_i .

The inverses nonlinear programming problem can be defined by determine the new coefficient of the objective function d_i which minimizes the objective function as follow:

$$\left.\begin{array}{c} \text{Min } z_{inv} = \sum_{i=1}^{n} |d_i - c_i|\, x_i \quad , i = 1,2,\dots n \\ \text{s.t.} \\ \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji} x_i \geq b_j \quad , \quad i = 1,2,\dots,n\,, j = 1,2\dots,m \\ x_i \geq 0 \\ l_i \leq x_i \leq u_i \end{array}\right] \quad (2)$$

Where d_i represent the coefficient of the objective function (i) and $|d_i - c_i|$ the norm of the difference between the current value and the new value of the coefficient. In 2010 Zhang and Xu formulate the new value of the coefficient of the decision variable in the objective function d_i as follow:

$$d_i = \frac{a_i^T \lambda^*}{\acute{f}_i(x_i^0)} \quad (3)$$

Where

a_i^T is the parameter of the decision variable in constraint, λ^* is the optimal Lagrange coefficient and $\acute{f}_i(x_i^0)$ is the first derivative for $f_i(x_i^0)$ at x^0 .

3. Algorithm

In this section, we represent the algorithm which formulated in 2010 by Zhang, J., & Liu, Z. as follows :-

Step (1): Solve the primal problem (1) using Lagrange method to find the Lagrange coefficient λ and the optimal solution which is$(\lambda^*, f(x_i^*), x_i^*)$.

Step (2): Examine the solution by using Border Hessian matrix to determine if the feasible point x^0 is minimum or maximum limit point.

Step (3): Find $a_i^T \lambda^*$ where

a_i^T is the parameter of the decision variable in constraint and

λ^* is the optimal Lagrange coefficient where x^0 is the feasible solution for the primal problem .

Step (4): Find the first derivative for $f_i(x_i^0)$ at x^0 which is $\acute{f}_i(x_i^0)$.

Step(5): Calculate the new value of the coefficient d_i in the objective function using the equation in (*3*) as the following:

$$d_i = \frac{a_i^T \lambda^*}{\acute{f}_i(x_i^0)}$$

Put the calculated value d_i from the above equation in the objective function to see if the value of the new objective function is the same as the value of the objective primal function or not. In the following section, we illustrate the previous section using two applied examples.

4. Applied example

Example (1)

In commercial institutions and banks the portfolio selection problem is one of the most important problems. Portfolio selection is defined as a choice of assets process to be annexed to the portfolio, and that is the most suitable and best way to make the portfolio better than other through fixed criterion. The decision makers want to arrive to the portfolio optimization through the following goals: maximize portfolio's expected return, minimize risks, maximize dividends and minimize the deviations from diversification goals. Assume the commercial bank wants to determine the optimal size of money to invest in each asset to balance between the return and risk. If the decision maker has n assets such that; $x_1$be the size of cash money, x_2 be the money invested in short term investments, x_3 be the money invested in long term investments, x_4 be the money invested for installment loans, x_5 be the money invested for commercial loans. Also, the decision maker has some constraints such that; the value of all assets equals the available money in the commercial bank for investment, the cash money should be increased of fixed percentage from the current accounts and fixed percentage from the saving accounts, and the size of money for each asset should be increased of a fixed percentage from the total available money. Consider the objective function is minimizing the risk that is measured by the covariance between the assets.

. Then, the portfolio selection problem can be formulated in the mathematical form as follows:-

$$\begin{aligned} &\text{Minimize } f(x) = \text{Min } X\sum X^T = \sum_{j=1}^{m}\sum_{i=1}^{n} x_j\sigma_{ij}x_i \\ &\text{S.T.} \\ &\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ji}x_i = b_j \\ &x_i \geq 0 \end{aligned} \quad (4)$$

Where $i = 1,2,\dots,n$, $j = 1,2,\dots,m$, a_{ji} is represent the coefficient of the decision variable x_i in the constraint j, b_j is represent the right hand side of the constraint (j), where $j = 1,2,\dots,m$,

x_i represent the size of money to invest in each asset to balance between the return and risk , σ_{ij} represent the covariance between x_i & x_j , where $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$.

For simplicity, Assume the director of portfolio selection problem wants to build a portfolio optimized for the client. Let's assume that he has two assets (A, B). A director is planning to invest so that requires a monthly return of these shares is not less than 0.03% and wishes to return from the stock should not be more than 0.075% of the budget.(assuming $\sigma_{AB} = 0$ where σ_{AB} is the covariance between A & B).

We can formulate this problem as follows :

$$\begin{aligned} &\text{minimize } f(x) = 2x_A^2 + 3x_B^2 \\ &\text{S.T.} \\ &2x_A + x_B \geq 3 \\ &3x_A + x_B \leq 75 \\ &x_A, x_B \geq 0 \end{aligned} \quad (5)$$

We can apply the steps in the algorithm in section 3 to solve the problem in (5) as the following:

Step (1): Solve the primal problem in (5) by using Lagrange method we find the optimal solution is:-

$$f(x)^* = 3.915 , x_A^* = 1.35 , x_B^* = 0.3 , \lambda_1^* = 0 , \lambda_2^* = 1.8$$

Step (2): Examine the solution using Bordered Hessian matrix H_B:

$$where$$

$$H_B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}_{2\times 2} \qquad Q = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$$

$$H_B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 6 \end{bmatrix}$$

$$H_B = 1 > 0$$

Since $H_B > 0$, then the function has minimum limit point.

Step (3): Assume the feasible solution is $x_a^0 = 1 , x_b^0 = 13.2$

Step (4): Find the first derivatives at $x_a^0 = 1$ and at $x_b^0 = 13.2$ as the following:

$$f'(1) = 4 , f'(13.2) = 79.2$$

Step (5): Find the new value of the coefficient d_i using the equation (3) as follows:

$$d_i = \frac{a_i^T \lambda^*}{f_i'(x_i^0)}$$

then $d_1 = 0, d_2 = 0.0227272$

The inverse optimization problem can be formulated as the following:

$$\left.\begin{aligned} &\text{Minimize } z_{inv} = 0.022x_B^2 \\ &\text{S.T.} \\ &2x_A + x_B \geq 3 \\ &3x_A + x_B \leq 75 \\ &x_A , x_B \geq 0 \end{aligned}\right] \quad (6)$$

The value of the objective function of the inverse problem at the feasible point $x_a^0 = 1, x_b^0 = 13.2$ is: $z_{inv} = 3.9$ which is the same value for the primal problem in step (1).

Example (2)

Several states are attempting to solve the problems of health care delivery by locating small satellite clinics in needy areas. The clinics are staffed by allied health personnel, who give primary treatment for minor routine care, meet emergency needs, offer health care advice, provide referral services for advanced treatments, and determine financial eligibility for assistance programmed. Transportation service is available at some clinics to bring patients to the clinic, and when necessary, take them to a central medical facility for access to more specialized and highly trained medical resources.

The location of such clinics is important if they are to be successful and achieve the maximum benefit. A number of factors influence the use of medical facilities. One is the distance that an individual must travel to accessibility. If we are concerned with improving the accessibility to medical resources for all members of population. Then one objective which may be pursued when locating a given number of clinics is to select sites which minimize the total travel required of the population in securing health care.

Our application reports a procedure that can be used to select clinic sites which will minimize the square of the distance d separating between coordinates of the satellite clinic location and suburban town ships. Suppose we have a Large health maintenance organization is planning to locate a satellite clinic in a location which is convenient to two suburban town ships ; and satellite clinic is a health care facility usually operated under the auspices of a large institution but situated in a location some distance from the larger health center. The decision maker wants to select a prelim-nary site by using the following criterion: determine the location which minimizes the sum of the square of the distances d separating between (x, y), where (x, y) is the given coordinates of the satellite clinic location and each town ship.

where (x, y) is the given coordinates of the satellite clinic location and each town ship. Also, the decision maker has some constraints such that; the coordinates of the satellite clinic location in fixed area. Then, problem to select the best location of the satellite clinic in general can be formulates as the following:

$$\begin{aligned} &\text{Minimize } d^2 = \text{Min } f(x,y) = \textstyle\sum_{i=1}^{n} (x - x_i)^2 + (y - y_i)^2 \\ &\text{S.T.} \\ &\textstyle\sum_{i=1}^{n} u_i x + \sum_{i=1}^{n} l_i y = b_i \\ &x_i, x, y_i, y \geq 0 \end{aligned} \qquad (7)$$

Where n refer to the number of suburban town ships $i = 1, 2, \dots, n$, u_i , l_i are represent the coefficients of the decision variables x , y in the constraint i , b_i is represent the right hand side of the constraint (i) where $i = 1, 2, \dots, n$, x_i , y_i represent the coordinates of the suburban town ships where $i = 1, 2, \dots, n$ and x , y are represent the coordinates of the satellite clinic location,d^2 represent the square of the distance separating these two points.

For simplicity, Assume the planner wants to locate a satellite clinic in a location which is convenient to two suburban town ships. Let's assume that he has two suburban town ships that have two coordinates $(10, -10), (-20, 10)$. The decision maker wants to determine the location which minimizes the sum of the square of the distances from each town ship to the satellite clinic and wishes to be the sum of the coordinates of the satellite clinic location are less than or equal 30.

The formulation of this problem can be as the following:

$$\text{Minimize } d = \text{Min } f(x,y) = [(x-10)^2 + (y+10)^2] + [(x+20)^2 + (y-10)^2]$$

$$\begin{aligned} &\text{Minimize } f(x,y) = 2x^2 + 20x + 2y^2 + 700 \\ &\text{S.T.} \\ &2x + y \leq 30 \\ &x,\ y \geq 0 \end{aligned} \qquad (8)$$

We can apply the steps in the algorithm in section 3 to solve the problem in (5) as the following:

Step (1): Solve the primal problem in (8) by using Lagrange method we find the optimal solution is:-

$$f^*(x,y) = 1290\,, x^* = 11\,, y^* = 8, \lambda^* = 32$$

Step (2): Examine the solution using Bordered Hessian matrixH_B:

$$where$$

$$H_B = \begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix}$$

$$P = [2 \quad 1]_{2*1} \qquad Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$H_B = \begin{bmatrix} 0 & P \\ P^T & Q \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$|H_B| = -2$$

Since $H_B < 0$, then the function has minimum limit point.

Step (3): Assume that our feasible solution is $x^0 = 15.75\,, y^0 = 10.46$

Step (4): Find the first derivatives at feasible points as the following:

$$\acute{f}_x(15.75) = 83 \;,\; \acute{f}_y(10.46) = 41.8$$

Step (5): Find the new value of the coefficient $d_i$as the following:

$$d_i = \frac{a_i^T \lambda^*}{\acute{f}_i(x_i^0)}$$

then $d_1 = 0.771$, $d_2 = 0.76$

The inverse optimization problem can be formulated as the following:-

$$\left.\begin{aligned} &\text{Min } Z_{inv} = 0.771x^2 + 20x + 0.76y^2 + 700 \\ &\text{S.T.} \\ &2x + y \le 30 \\ &x, y \ \ge 0 \end{aligned}\right] \quad (9)$$

The value of the objective function of the inverse problem at the feasible point $x^0 = 15.75$, $y^0 = 10.46$ is $z_{inv} = 1290$. We find that the value of z_{inv} is the same value of our primal nonlinear separable optimization problem.

5. Conclusion

The inverse nonlinear convex programming problem with linear constraints has many important applications in different areas. We introduce the algorithm to solve the inverse nonlinear convex problem with linear constraints. Two applied examples are considered . In Example (1), a portfolio optimization is presented because of its importance in industrial area, Decision maker plans to determine the optimal size of money to invest in each asset to make a balance between the return and risk. Also, in Example (2) , a decision maker plans to locate a satellite clinic in a location to be convenient to some suburban town ships; he needs to select a prelim-nary site by minimizing the sum of the square of the distances from each town ship to the satellite clinic. From these two examples, we can reach to that the role of the inverse problem which comes to enable decision makers to turn a possible solution to an optimal solution are while maintaining the current optimal value of the objective function points in the primal function .

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