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Aliasing free for stable random field

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Abstract

This paper presents a discrete estimation for the spectral density function of a stable random field with continuous time from observations taken at discrete instants of time. Under the condition on spectral density that it has a compact support, asymptotic expressions for the bias and variance are derived.

1 Introduction

A process $\{X_t\}_t$ is called multidimensional or random field when the parameter t which indexes its values has several components $t_1, \ldots, t_p$, say. In this case the parameter t is vector valued and of course it can no longer represent time.
If we observe the heights of a sea wave at different points in a given area (at a fixed time instant) then we have a process, $X_{x,y}$, which depends on two spatial parameters x and y. If we record the heights at different points over an interval of time then we have a process $X_{x,y,t}$ which depends on two spatial parameters x and y, and a time parameter t.
Multidimensional processes arises naturally when we consider fields and wish to study their spatial as well as their temporal variations. Thus, in the statistical theory of turbulence each component of the velocity vector may be regarded as a four dimensional process, depending on three spatial coordinates and one time coordinate (cf. Bartlett (1955), p.193), while Longuett-Higgins (1957) used a two-dimensional process to describe the behaviour

of sea waves, as indicated above. Pierson and Tick (1957) considered two-dimensional processes in metrology and oceanography, and a similar model has been suggested for the analysis of waves in a paper mill.
Other models for two-dimensional processes have discussed by Whittle (1954), Walker and Young (1955) and Heine (1955).
In this paper,we consider a complex stationary symmetric α stable continuous time random field $X = \{X(t_1, t_2)/t_1, t_2 \in R\}$ where the parameter $\alpha \in (0, 2)$ is assumed known; more specifically, X is a complex-valued stochastic process for wich the finite dimensional characteristic function is:

$$Ee^{\left(i\Re e \sum_{j=1}^{n} z_j X(t_{1j}, t_{2j})\right)} = e^{\left(-C_\alpha \int_{-\infty}^{\infty} \left|\sum_{j=1}^{n} z_j e^{i(t_{1j}u_1 + t_{2j}u_2)}\right|^\alpha \phi(u_1, u_2) du_1 du_2\right)}$$

with $C_\alpha = (\alpha\pi)^{-1} \int_0^\pi |\cos(\theta)|^\alpha d\theta$, where ϕ is a nonnegative integrable function called the spectral density of the process X. This spectral density plays a role analogous to that played by the usual power spectral density function of a second order stationary process. It is clear that the spectral density ϕ fully describes the distribution of the process X. Alternatively X has the integral representation:

$$X(t_1, t_2) = \int_{\mathbb{R}^2} \exp\left[i(\lambda_1 t_1 + \lambda_2 t_2)\right] d\xi(\lambda_1, \lambda_2), \qquad (1)$$

where ξ is (S.α.S) process with independent isotropic increments; that means ξ is an additive complex function defined on the Borel subsets of $\mathbb{R}^2$, such that :

- for any integer k, any Borel sets $B_1, B_2, \cdots, B_k$, the random vector: $(\xi(B_1), \xi(B_2), \cdots, \xi(B_k))$ is (S.α.S),
- for any integer k, any disjoint Borel sets $B_1, B_2, \cdots, B_k$, the complex (S.α.S) random variables $\xi(B_1), \xi(B_2), \cdots, \xi(B_k)$ are independent,
- for all Borel sets B, the distribution of the random variable $e^{i\theta}\xi(B)$ is independent of θ.

The stochastic integral (1) is defined by means of convergence in probability for more detail see Cambanis (1983), Masry and Cambanis (1984) and Samorodnitsky and Taqqu (1993). The spectral density function is already estimated by Sabre(1995), In the case when the process random field X has discrete time. For the continuous time random field defined in (1), we can give, as Masry and combanis (1984), an estimate of the spectral density by observing the process on an interval continuous of time $[-T_1, T_2] \times [-T_2, T_2]$.Our work is motived by the fact that, in practice, it is not obvious to observe the process on continuous interval of time. Our goal is to establish nonparametric estimate for the spectral density ϕ of X, sampled at instants $\left(t_n, t'_m\right)$, where the sampling instants t_n and t'_m are equally spaced, i.e., $t_n = n\tau_1$, $t'_m = m\tau_2$, $\tau_1, \tau_2 > 0$, it is known that aliasing of ϕ occurs. For more details about aliasing phenomenon see Masry (1978). To avoid this difficulty, we suppose that the spectral density ϕ is vanishing outside the interval $[-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2]$, where Ω_1 and Ω_2 are two nonnegative real numbers. We introduce an estimate depending on Ω_1 and Ω_2. We show that it is asymptotically unbiased estimate but not consistent (theorem 3.1).However by smoothing it via spectral window, we show that its mean-square consistency as an estimate, along with rates of convergence is established (theorems 4.1, 4.2, 4.3 and 4.4). This paper is organized as follows: In the second section we give some basic definition and present two lemmas and inequalities which are used in de sequel. The third section provides the theorem 3.1. we give an estimator asymptotically unbiased but not consistent. We smooth this estimator, in section 4, by two spectral The fourth section provides the theorems 4.1, 4.2 and 4.3.

2 Preliminaries

We introduce some basic notation and properties used throughout the paper. A real random variable Y is symmetric α-stable (S.α.S), $0 < \alpha < 2$, if $E \exp\{irY\} = \exp\{-c_Y |r|^\alpha\}$ for all r and some $c_Y \geq 0$. The random variables $Y_1, ..., Y_n$ are jointly (S.α.S) if all linear combinations $a_1Y_1 + a_2Y_2 + ... + a_nY_n$ are (S.α.S). A random complex variable, $Y = Y_1 + iY_2$ is (S.α.S) if Y_1, Y_2 are jointly (S.α.S).

A stochastic process $\{X(t), \ -\infty < t < \infty\}$ is called a (S.α.S) process if every finite linear combination $Y = \sum_{i=1}^{n} a_i X(t_i)$ has a (S.α.S) distribution.

As in (Demesh(1988); Sabre(1994,1995,1999), we give the definition of the Jackson polynomial kernel.

Definition 2.1

The following function is called Jackson polynomial

$$H^{(N)}(l) = \frac{1}{q_{k,n}} \left(\frac{\sin(\frac{nl}{2})}{\sin(\frac{l}{2})} \right)^{2k} \quad \text{where} \quad q_{k,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin(\frac{nl}{2})}{\sin(\frac{l}{2})} \right)^{2k} dl,$$

N is a fixed real number such that: $N = 2k(n-1)+1$, where $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \left\{\frac{1}{2}\right\}$; and if $k = \frac{1}{2}$ then n is an odd integer.
In section 5, we show that it exists a function h_k satisfying:

$$H^{(N)}(\lambda) = \sum_{m'=-k(n-1)}^{k(n-1)} h_k(m'/n) \cos(\lambda m'), .$$

from the Jackson polynomial function, we define the following kernel which is used in the construction of the periodogram.

$$|H_N(\lambda)|^\alpha = \left|A_N H^{(N)}(\lambda)\right|^\alpha \quad \text{where} \quad A_N = \left(\int_{-\pi}^{\pi} \left|H^{(N)}(\lambda)\right|^\alpha d\lambda \right)^{\frac{-1}{\alpha}}.$$

If $k = \frac{1}{2}$, this kernel coincides with the kernel defined by Hosoya (1978).

We state an important technical result in the following lemma which proof is given in sabre (1995,1999).

Lemma 2.1

Let $B'_{\alpha,N}$ and $J_{N,\alpha}$ be the following integrals:

$$B'_{\alpha,N} = \int_{-\pi}^{\pi} \left| \frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right|^{2k\alpha} d\lambda \quad \text{and} \quad J_{N,\alpha} = \int_{-\pi}^{\pi} |u|^\gamma |H_N(\lambda)|^\alpha d\lambda,$$

where $\gamma \in]0, 2]$. Then

$$B'_{\alpha,N} \begin{cases} \geq 2\pi \left(\frac{2}{\pi}\right)^{2k\alpha} n^{2k\alpha-1} & \text{if } 0 < \alpha < 2, \\ \leq \dfrac{4\pi k\alpha}{2k\alpha - 1} n^{2k\alpha-1} & \text{if } \dfrac{1}{2k} < \alpha < 2, \end{cases}$$

$$J_{N,\alpha} \leq \begin{cases} \dfrac{\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma - 2k\alpha + 1)} \dfrac{1}{n^{2k\alpha-1}} & \text{if } \dfrac{1}{2k} < \alpha < \dfrac{\gamma+1}{2k}, \\ \dfrac{2k\alpha\pi^{\gamma+2k\alpha}}{2^{\gamma+2k\alpha}(\gamma+1)(2k\alpha - \gamma - 1)} \dfrac{1}{n^\gamma} & \text{if } \dfrac{\gamma+1}{2k} < \alpha < 2, \end{cases}$$

Lemma 2.2

If ξ is a (S.α.S) process with independent and isotropic increments, then for every $f \in L_\alpha(\mu)$, we have:

$$\mathbf{E}\left\{\exp\left(i\mathcal{R}e\left[\int_{\mathbf{R}^2} f(u_1,u_2)d\xi(u_1,u_2)\right]\right)\right\} = \exp\left(-C_\alpha \int_{\mathbf{R}^2} |f(u_1,u_2)|^\alpha d\mu(u_1,u_2)\right),$$

where $C_\alpha = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |\cos(\theta)|^\alpha d\theta$.

The proof of this lemma is similarly to classical result in the unidimensional case proved by Cambanis (1983).

In the following, we list some inequalities used in the sequel which are proved by (Masry and Cambanis, (1984); Sabre (1995):

For all real x, y and $0 < \alpha \leq 2$, we have:

$$||x+y|^\alpha - |x|^\alpha - |y|^\alpha| \leq 2\,|xy|^{\frac{\alpha}{2}}, \tag{2}$$

For all real $x, y \geq 0$ and $r > 2$, we have:

$$|x^r - y^r| \leq \frac{r}{2}\left(x^{r-1} + y^{r-1}\right)|x-y| \tag{3}$$

For all real $x, y \geq 0$ and $0 < r < 1$, we have:

$$|x^r - y^r| \leq |x-y|^r. \tag{4}$$

For all real x, y and $r > 0$, we have:

$$|x+y|^r \leq 2^r\left(|x|^r + |y|^r\right). \tag{5}$$

We give a hypothesis of the regularity on ϕ which will be used to improve the rate of convergence of the estimator:

$$|\phi(\lambda_1 + u_1, \lambda_2 + u_2) - \phi(\lambda_1, \lambda_2)| \leq C_1||(u_1,u_2)||^\gamma \qquad \text{where } 0 < \gamma < 1 \quad (\mathcal{H})$$

and C_1 is a nonnegative constant.

3 The periodogram

In this section we give a periodogram and we develop its proprieties. Assume that the process X defined in (1) is observed at instants $t_j = j\tau_1$ and $t_\ell = \ell\tau_2$, $j = 0,1,\ldots,N-1$, $\ell = 0,1,\ldots,M-1$, with $\tau_1 = \frac{2\pi}{\omega_1}$ and $\tau_2 = \frac{2\pi}{\omega_2}$, where ω_i is a real number strictly greater than $2\Omega_i$ for $i = 1,2$. We define the periodogram $\hat{I}_{N,M}$ on $]-\Omega_1,\Omega_1[\times]-\Omega_2,\Omega_2[$ as follows:

$$\hat{I}_{N,M}(\lambda_1,\lambda_2) = C_{p,\alpha}|I_{N,M}(\lambda_1,\lambda_2)|^p, \qquad 0 < p < \alpha/2$$

where

$$\begin{aligned} I_{N,M}(\lambda_1,\lambda_2) &= [\tau_1\tau_2]^{\frac{1}{\alpha}} A_N A_M \mathcal{R}e\Bigg[\sum_{n'=-k(n-1)}^{n'=k(n-1)} \sum_{m'=-k(m-1)}^{m'=k(m-1)} h_k\left(\frac{n'}{n}\right) h_k\left(\frac{m'}{m}\right) \times \\ & e^{(-i(n'\tau_1\lambda_1+m'\tau_2\lambda_2))} X\Big(n'\tau_1 + k(n-1)\tau_1\ ,\ m'\tau_2 + k(m-1)\tau_2\Big)\Bigg], \end{aligned}$$

and the normalization constant $C_{p,\alpha}$ is given by $C_{p,\alpha} = \frac{D_p}{F_{p,\alpha}[C_\alpha]^{p/\alpha}}$, with $D_p = \int_{-\infty}^{\infty} \frac{1-\cos(u)}{|u|^{1+p}}du$; $F_{p,\alpha} = \int_{-\infty}^{\infty} \frac{1-e^{-|u|^\alpha}}{|u|^{1+p}}du$ and C_α is given in the lemma 2.2.

Lemma 3.1 *The characteristic function of $I_{N,M}$, $\mathbb{E}\exp[irI_{N,M}(\lambda_1,\lambda_2)]$ converges to* $\exp\Big[-C_\alpha|r|^\alpha\phi(\lambda_1,\lambda_2)\Big]$.

Proof. By substituting (1) in the expression of $I_{N,M}$, we have:

$$\begin{aligned} I_{N,M}(\lambda_1,\lambda_2) &= [\tau_1\tau_2]^{\frac{1}{\alpha}} A_N A_M \mathcal{R}e \int_{\mathbb{R}^2} \sum_{n'=-k(n-1)}^{n'=k(n-1)} \sum_{m'=-k(m-1)}^{m'=k(m-1)} h_k\left(\frac{n'}{n}\right) h_k\left(\frac{m'}{m}\right) \\ & \times \exp\Big\{i\Big[n'\tau_1(\lambda_1-u_1) + m'\tau_2(\lambda_2-u_2)\Big]\Big\} \\ & \exp\Big\{i\Big[\tau_1 u_1 k(n-1) + \tau_2 u_2 k(m-1)\Big]\Big\} d\xi(u_1,u_2) \end{aligned}$$

It follows from the lemma 2.2 and the definition of the Jackson polynomial kernel that the characteristic function is the form:

$$\mathbb{E}\exp[irI_{N_M}(\lambda_1,\lambda_2)] = \exp[-C_\alpha|r|^\alpha\psi_{N,M}(\lambda_1,\lambda_2)]. \qquad (6)$$

where

$$\psi_{N,M}(\lambda_1,\lambda_2)=\sum_{j,j'\in\mathbf{Z}}\int_{(2j-1)\pi}^{(2j+1)\pi}\int_{(2j'-1)\pi}^{(2j'+1)\pi}Z(v_1,v_2)dv_1dv_2 \quad (7)$$

with $Z(v_1,v_2)=\left|H_N(v_1-\tau_1\lambda_1)\,H_M(v_2-\tau_2\lambda_2)\right|^{\alpha}\phi\left(\frac{v_1}{\tau_1},\frac{v_2}{\tau_2}\right)$. Let $v_1=y_1-2\pi j$ and $v_2=y_2-2\pi j'$. Since H_N and H_M are 2π-periodic, we obtain

$$\psi_{N,M}(\lambda_1,\lambda_2)=\sum_{j,j'\in\mathbf{Z}}\iint_{-\pi}^{\pi}\left|H_N(y_1-\tau_1\lambda_1)\,H_M(y_2-\tau_2\lambda_2)\right|^{\alpha}\phi_{j,j'}(y_1,y_2)dy_1dy_2,$$

where $\phi_{j,j'}(y_1,y_2)=\phi\left(\frac{y_1}{\tau_1}-\frac{2\pi}{\tau_1}j\,,\,\frac{y_2}{\tau_2}-\frac{2\pi}{\tau_2}j'\right)$.

Let j be an integer such that $-\Omega_1<\frac{y_1-2\pi j}{\tau_1}<\Omega_1$. Using the fact that $\tau_i\Omega_i<\pi$ and $|y_1|<\pi$, we get $|j|<\frac{\tau_1\Omega_1}{2\pi}+\frac{1}{2}<1$ and then $j=0$. Therefore

$$\psi_{N,M}(\lambda_1,\lambda_2)=\iint_{-\pi}^{\pi}\left|H_N(y_1-\tau_1\lambda_1)\,H_M(y_2-\tau_2\lambda_2)\right|^{\alpha}\phi\left(\frac{y_1}{\tau_1},\frac{y_2}{\tau_2}\right)dy_1dy_2. \quad (8)$$

Since ϕ is continuous and $|H_N|^{\alpha}$, $|H_M|^{\alpha}$ are two kernels, the result follows.

Theorem 3.1 *Let $-\Omega_1<\lambda_1<\Omega_1$; $-\Omega_2<\lambda_2<\Omega_2$ then $\hat{I}_{N,M}(\lambda_1,\lambda_2)$ is asymptotically unbiased estimate of $\phi(\lambda_1,\lambda_2)^{(p/\alpha)}$ but not consistent*

$$\begin{aligned}\lim_{N,M\to\infty}\mathbf{E}\left[\hat{I}_{N,M}(\lambda_1,\lambda_2)\right] &= [\phi(\lambda_1,\lambda_2)]^{\frac{p}{\alpha}},\\ \lim_{N,M\to\infty}\operatorname{var}\left[\hat{I}_{N,M}(\lambda_1,\lambda_2)\right] &= V_{p,\alpha}[\phi(\lambda_1,\lambda_2)]^{\frac{2p}{\alpha}}.\end{aligned}$$

where $V_{p,\alpha}=C_{p,\alpha}^2C_{2p,\alpha}^{-1}-1$.

Proof. As in Masry and Cambanis(1984), we use the following equality: for all real x and $0<p<\alpha/2$,

$$|x|^p=D_p^{-1}\int_{-\infty}^{\infty}\frac{1-\cos(xu)}{|u|^{1+p}}du=D_p^{-1}\mathcal{R}\mathrm{e}\int_{-\infty}^{\infty}\frac{1-e^{ixu}}{|u|^{1+p}}du. \quad (9)$$

Replacing x by $I_{N,M}$, we obtain

$$\hat{I}_{N,M}(\lambda_1,\lambda_2)=\frac{1}{F_{p,\alpha}[C_\alpha]^{p/\alpha}}\mathcal{R}\mathrm{e}\int_{-\infty}^{\infty}\frac{1-\exp\{iuI_{N,M}(\lambda_1,\lambda_2)\}}{|u|^{1+p}}du, \quad (10)$$

Using (6) and the definition of the $F_{p,\alpha}$, we get

$$\mathbf{E}\hat{I}_{N,M}(\lambda_1,\lambda_2) = [\psi_{N,M}(\lambda_1,\lambda_2)]^{p/\alpha}. \tag{11}$$

Since $\psi_{N,M}(\lambda_1,\lambda_2)$ converges to $\phi(\lambda_1,\lambda_2)$, $\hat{I}_{N,M}(\lambda_1,\lambda_2)$ is an asymptotically unbiased estimate of $[\phi(\lambda_1,\lambda_2)]^{\frac{p}{\alpha}}$, and from (9) it follows that

$$\mathbf{E}\left(\hat{I}_{N,M}(\lambda_1,\lambda_2)\right)^2 = C_{p,\alpha}^2 D_{2p}^{-1} C_{\alpha}^{\frac{2p}{\alpha}} [\psi_{N,M}(\lambda_1,\lambda_2)]^{2p/\alpha}. \tag{12}$$

Hence, from (10) and (11), $\mathrm{var}\left[\hat{I}_{N,M}(\lambda_1,\lambda_2)\right] = V_{\alpha,p}[\psi_{N,M}(\lambda_1,\lambda_2)]^{\frac{2p}{\alpha}}$. Thus the asymptotic variance of $\hat{I}_{N,M}(\lambda_1,\lambda_2)$ is proportional to $[\phi(\lambda_1,\lambda_2)]^{\frac{2p}{\alpha}}$.

4 Smoothing the Periodogram

In order to obtain a consistent estimate of $[\phi(\lambda_1,\lambda_2)]^{\frac{p}{\alpha}}$, we smooth the periodogram via two spectral windows W_N and W_M defined by: $W_N(v) = M_N W(M_N v)$; $W_M(v) = L_M W(L_M v)$ where M_N and L_M satisfies:

$$\lim_{N\to+\infty} M_N = \infty; \lim_{M\to+\infty} L_M = \infty \text{ and } \lim_{N\to+\infty} \frac{M_N}{N} = 0; \lim_{M\to+\infty} \frac{L_M}{M} = 0,$$

where W is a nonnegative, even, continuous function , vanishing for $|\lambda| > 1$ such that $\int_{-1}^{1} W(u)du = 1$. The bandwidths of spectral windows are then respectively proportional to $1/M_N$ and $1/L_M$. Rachdi and sabre (1998) give a criterion to choice the spectral bandwidth for random field by using the cross validation method. We consider the smooth periodogram $f_{N,M}$ defined by:

$$f_{N,M}(\lambda_1,\lambda_2) = \int_{\mathbf{R}^2} W_N(\lambda_1-u_1)W_M(\lambda_2-u_2)\hat{I}_{N,M}(u_1,u_2)du_1du_2,$$

$-\Omega_1 < \lambda_1 < \Omega_1$ and $-\Omega_2 < \lambda_2 < \Omega_2$

We first show that $f_{N,M}(\lambda_1,\lambda_2)$ is an asymptotically unbiased estimator of $[\phi(\lambda_1,\lambda_2)]^{\frac{p}{\alpha}}$ for $-\Omega_1 < \lambda_1 < \Omega_1$ and $-\Omega_2 < \lambda_2 < \Omega_2$.

Theorem 4.1 *Let* $-\Omega_1 < \lambda_1 < \Omega_1$; $-\Omega_2 < \lambda_2 < \Omega_2$, *then*

$$\mathbf{E}\left[f_{N,M}(\lambda_1,\lambda_2)\right] - [\phi(\lambda_1,\lambda_2)]^{\frac{p}{\alpha}} = o(1).$$

if ϕ satisfies the hypothesis $\mathcal{H}$ with $\gamma < 2k\alpha - 1$, then

$$\mathbf{E}\left[f_{N,M}(\lambda_1, \lambda_2)\right] - \left[\phi(\lambda_1, \lambda_2)\right]^{\frac{p}{\alpha}} = O\left(T_N(\lambda_1) + T_M(\lambda_2) + \frac{1}{M_N^{\gamma}} + \frac{1}{L_M^{\gamma}}\right)$$

where

$$T_N = \begin{cases} \left(\frac{1}{n^{2k\alpha-1}}\right) & \text{if } \lambda_1 \neq 0 \\ \left(\frac{1}{M_N n^{2k\alpha-1}}\right) & \text{if } \lambda_1 = 0 \end{cases} \quad \text{and } T_M = \begin{cases} \left(\frac{1}{m^{2k\alpha-1}}\right) & \text{if } \lambda_2 \neq 0 \\ \left(\frac{1}{L_M m^{2k\alpha-1}}\right) & \text{if } \lambda_2 = 0 \end{cases}$$

Proof. By the definition of the spectral window we have:
$\mathbf{E}\left[f_{N,M}(\lambda_1, \lambda_2)\right] =$
$\int_{\mathbb{R}^2} M_N W\left[M_N(\lambda_1 - u_1)\right] L_M W\left[L_M(\lambda_2 - u_2)\right] \mathbf{E}\left[\hat{I}_{N,M}(u_1, u_2)\right] du_1 du_2.$
Let $M_N(\lambda_1 - u_1) = v_1$ and $L_M(\lambda_2 - u_2) = v_2$ and from (11), we obtain:

$$\mathbf{E}[f_{N,M}(\lambda_1, \lambda_2)] = \iint_{-1}^{1} W(v_1)W(v_2)\left[\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right)\right]^{\frac{p}{\alpha}} dv_1 dv_2. \quad (13)$$

Using the fact that $\int_{-1}^{1} W(u)du = 1$ and the inequality (4), we get:

$\left|\mathbf{E}\left[f_{N,M}(\lambda_1, \lambda_2)\right] - \left[\phi(\lambda_1, \lambda_2)\right]^{\frac{p}{\alpha}}\right|$
$\leq \int_{-1}^{1}\int_{-1}^{1} W(v_1)W(v_2)\left|\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right) - \phi(\lambda_1, \lambda_2)\right|^{\frac{p}{\alpha}} dv_1 dv_2.$

We now examine the limit of $\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right)$ as $N, M \to \infty$. From (7) we get: $\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right) =$

$$\sum_{j,j' \in \mathbb{Z}} \int_{(2j-1)\pi}^{(2j+1)\pi} \int_{(2j'-1)\pi}^{(2j'+1)\pi} |\rho_{N,M}(u_1, u_2)|^{\alpha} \phi\left(\frac{u_1}{\tau_1}, \frac{u_2}{\tau_2}\right) du_1 du_2, \quad (14)$$

where

$$\rho_{N,M}(u_1, u_2) = H_N\left(u_1 - \tau_1\left(\lambda_1 - \frac{v_1}{M_N}\right)\right) H_M\left(u_2 - \tau_2\left(\lambda_2 - \frac{v_2}{L_M}\right)\right)$$

Let $2j\pi - \tau_1\left(\lambda_1 - \frac{v_1}{M_N}\right) + u_1 = s_1$ and $2j'\pi - \tau_2\left(\lambda_2 - \frac{v_2}{L_M}\right) + u_2 = s_2$ we obtain:

$$\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right) = \sum_{j,j' \in \mathbb{Z}} \iint_{-\pi}^{+\pi} |H_N(s_1)H_M(s_2)|^\alpha R_{j,j'}(s_1, s_2, v_1, v_2) ds_1 ds_2.$$

where

$$R_{j,j'}(s_1, s_2, v_1, v_2) = \phi\left(\lambda_1 - \frac{v_1}{M_N} - \frac{s_1}{\tau_1} + \frac{2\pi}{\tau_1}j\ ,\ \lambda_2 - \frac{v_2}{L_M} - \frac{s_2}{\tau_2} + \frac{2\pi}{\tau_2}j'\right).$$

Since the function ϕ is uniformly continuous on $[-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2]$ and the fact that $|H_N|^\alpha$, $|H_M|^\alpha$ are two kernels, $\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right)$ converges to $\sum_{j,j' \in \mathbb{Z}} \phi\left(\lambda_1 + \frac{2\pi j}{\tau_1} + \lambda_2 + \frac{2\pi j'}{\tau_2}\right)$.

Let j and j' be two integers such that $-\Omega_1 < \frac{\tau_1\lambda_1 + 2\pi j}{\tau_1} < \Omega_1$ and $-\Omega_2 < \frac{\tau_2\lambda_2 + 2\pi j'}{\tau_2} < \Omega_2$. The definition of τ_i implies that $|\tau_i\lambda_i| < |\tau_i\Omega_i| < \pi$. It is easy to see that $|j| < 1$ and $|j'| < 1$ and then $j = j' = 0$. Thus we obtain $\mathbf{E}\left[f_{N,M}(\lambda_1, \lambda_2)\right] - [\phi(\lambda_1, \lambda_2)]^{\frac{p}{\alpha}} = o(1)$.

The rate of convergence:

We assume that the spectral density ϕ satisfies the hypothesis $\mathcal{H}$. Denote by $F \triangleq |bias(f_{N,M}(\lambda_1, \lambda_2))| \triangleq \left|\mathbf{E}[f_{N,M}(\lambda_1, \lambda_2)] - [\phi(\lambda_1, \lambda_1)]^{p/\alpha}\right|$. It follows from the inequality (3) below that

$$\begin{aligned} F \ \leq\ & \frac{p}{2\alpha}\iint_{-1}^{1} W(v_1)W(v_2)\left[\left[\psi_{N,M}\left(\lambda_1 - \tfrac{v_1}{M_N}, \lambda_2 - \tfrac{v_2}{L_M}\right)\right]^{\frac{p}{\alpha}-1} + [\phi(\lambda_1, \lambda_2)]^{\frac{p}{\alpha}-1}\right] \\ & \times\left|\psi_{N,M}\left(\lambda_1 - \tfrac{v_1}{M_N}, \lambda_2 - \tfrac{v_2}{L_M}\right) - \phi(\lambda_1, \lambda_2)\right| dv_1 dv_2 \end{aligned}$$

Since $\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right)$ converges to $\phi(\lambda_1, \lambda_2)$, getting the rate of the convergence for F, requires to examine the rate of convergence for

$$\int\int_{-1}^{1} W(v_1)W(v_2)\left|\psi_{N,M}\left(\lambda_1 - \frac{v_1}{M_N}, \lambda_2 - \frac{v_2}{L_M}\right) - \phi(\lambda_1, \lambda_2)\right| dv_1 dv_2.$$

Indeed, from (8), we obtain

$$\psi_{N,M}\left(\lambda_1 - \tfrac{v_1}{M_N}, \lambda_2 - \tfrac{v_2}{L_M}\right)$$
$$= \iint_{-\pi}^{\pi} \left| H_N\left(y_1 - \tau_1\lambda_1 + \tfrac{\tau_1 v_1}{M_N}\right) H_M\left(y_2 - \tau_2\lambda_2 + \tfrac{\tau_2 v_2}{L_M}\right)\right|^{\alpha} \phi\left(\tfrac{y_1}{\tau_1}, \tfrac{y_2}{\tau_2}\right) dy_1 dy_2$$

Denote by $\Delta\left(\psi_{N,M}, \phi\right) = \psi_{N,M}\left(\lambda_1 - \tfrac{v_1}{M_N}, \lambda_2 - \tfrac{v_2}{L_M}\right) - \phi(\lambda_1, \lambda_2)$. Putting $t = -\left(y_1 - \tau_1\lambda_1 + \tfrac{\tau_1 v_1}{M_N}\right)$ and $t' = -\left(y_2 - \tau_2\lambda_2 + \tfrac{\tau_2 v_2}{L_M}\right)$, using the hypothesis $\mathcal{H}$, we obtain

$$\begin{aligned}
|\Delta(\psi_{N,M}, \phi)| \leq{}& C_1 \int_{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} - \pi}^{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} + \pi} \int_{\tau_2\lambda_2 - \frac{\tau_2 v_2}{L_M} - \pi}^{\lambda_2 - \frac{\tau_2 v_2}{L_M} + \pi} \left| H_N(t) H_M(t') \right|^{\alpha} \\
&\times \left\| \left(-\tfrac{v_1}{M_N} - \tfrac{t}{M_N}, -\tfrac{v_2}{L_M} - \tfrac{t'}{L_M} \right) \right\|^{\gamma} dt dt'
\end{aligned}$$

The inequality (5) implies that

$$\begin{aligned}
|\Delta(\psi_{N,M}, \phi)| \leq{}& 2^{2\gamma} C_1 \int_{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} - \pi}^{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} + \pi} |H_N(t)|^{\alpha} \left(\left| \frac{u}{M_N} \right|^{\gamma} + \left| \frac{t}{\tau_1} \right|^{\gamma} \right) dt \\
&+ 2^{2\gamma} C_1 \int_{\tau_2\lambda_2 - \frac{\tau_2 v_2}{L_M} - \pi}^{\tau_2\lambda_2 - \frac{\tau_2 v_2}{L_M} + \pi} |H_M(t')|^{\alpha} \left(\left| \frac{v}{L_M} \right|^{\gamma} + \left| \frac{t'}{\tau_2} \right|^{\gamma} \right) dt' \\
\leq{}& 2^{2\gamma} C_1 \left| \tfrac{u}{M_N} \right|^{\gamma} + 2^{2\gamma} \tfrac{C_1}{\tau_1^{\gamma}} \int_{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} - \pi}^{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} + \pi} |H_N(t)|^{\alpha} |t|^{\gamma} dt \\
&+ 2^{2\gamma} \tfrac{C_1}{\tau_2^{\gamma}} \left| \tfrac{v}{L_M} \right|^{\gamma} + 2^{2\gamma} C_1 \int_{\tau_2\lambda_2 - \frac{\tau_2 v_2}{L_M} - \pi}^{\tau_2\lambda_2 - \frac{\tau_2 v_2}{L_M} + \pi} |H_M(t')|^{\alpha} |t'|^{\gamma} dt' \qquad (**)
\end{aligned}$$

The first integral of (**) is bounded as follows:

$$\begin{aligned}
\int_{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} - \pi}^{\tau_1\lambda_1 - \frac{\tau_1 v_1}{M_N} + \pi} |H_N(t)|^{\alpha} |t|^{\gamma} dt \leq{}& \int_{-|\tau_1\lambda_1| - |\frac{\tau_1 v_1}{M_N}| - \pi}^{-\pi} |H_N(t)|^{\alpha} |t|^{\gamma} dt \\
&+ \int_{-\pi}^{\pi} |H_N(t)|^{\alpha} |t|^{\gamma} dt \\
&+ \int_{\pi}^{|\tau_1\lambda_1| + |\frac{\tau_1 v_1}{M_N}| + \pi} |H_N(t)|^{\alpha} |t|^{\gamma} dt. \qquad (15)
\end{aligned}$$

The function $|H_N(.)|^\alpha$ is even, then the first and the last integrals in the right hand side of the above inequality are equal. Since $\frac{\tau_1 v_1}{M_N}$ converges to zero and $\tau_1\lambda_1 < \tau_1\Omega_1 < \pi$, for a large N we have

$$\begin{aligned}\int_\pi^{|\tau_1\lambda_1|+|\frac{\tau_1 v_1}{M_N}|+\pi} |H_N(t)|^\alpha |t|^\gamma dt &\leq (2\pi)^\gamma \int_\pi^{|\tau_1\lambda_1|+|\frac{\tau_1 v_1}{M_N}|+\pi} |H_N(t)|^\alpha dt \\ &\leq \frac{(2\pi)^\gamma}{B'_{\alpha,N}} \frac{|\tau_1\lambda_1|+\frac{\tau_1}{M_N}}{\left|\sin\left(\frac{\pi+|\tau_1\lambda_1|+\frac{\tau_1}{M_N}}{2}\right)\right|^{2k\alpha}}\end{aligned}$$

From the lemma 2.1, we obtain,

$$\int_\pi^{|\tau_1\lambda_1|+|\frac{\tau_1 v_1}{M_N}|+\pi} |H_N(t)|^\alpha |t|^\gamma dt = T_N(\lambda_1)$$

where $T_N(\lambda_1)$ is defined in the theorem 4.1. From the lemma 2.1, the second integral in the right hand side of (15) is bounded. By using the same way for the second integral of (**), the result follows readily.

Theorem 4.2 *Let* $-\Omega_1 < \lambda_1 < \Omega_1$ *and* $-\Omega_2 < \lambda_2 < \Omega_2$ *such as* $\phi(\lambda_1, \lambda_2) > 0$. *Then* $\mathrm{var}\left[f_{(N,M)}(\lambda_1, \lambda_2)\right]$ *converges to zero. If* $M_N = n^c$, $L_M = m^{c'}$ *with* $\frac{1}{2k^2\alpha^2} < c < \frac{1}{2}$ *and* $\frac{1}{2k^2\alpha^2} < c' < \frac{1}{2}$ *then*

$$\mathrm{var}\left[f_{N,M}(\lambda_1, \lambda_2))\right] = O\left(\frac{1}{n^{1-2c}} \frac{1}{m^{1-2c'}}\right)$$

Proof. It is clear that the variance of $f_{(N,M)}(\lambda_1, \lambda_2)$ can be written as follows :

$$\begin{aligned}\mathrm{var}[f_{N,M}(\lambda_1, \lambda_2)] &= \int_{\mathbb{R}^4} W_N(\lambda_1 - u_1)W_M(\lambda_2 - u_2)W_N(\lambda_1 - u'_1)W_M(\lambda_2 - u'_2) \\ &\times \ \mathrm{cov}\left[\hat{I}_{N,M}(u_1, u_2), \hat{I}_{N,M}(u'_1, u'_2)\right] du_1 du_2 du'_1 du'_2.\end{aligned}$$

Let $x_1 = M_N(\lambda_1 - u_1)$; $x'_1 = M_N\left(\lambda_1 - u'_1\right)$ and $x_2 = L_M(\lambda_2 - u_2)$; $x'_2 = L_M\left(\lambda_2 - u'_2\right)$. By using the fact that W is zero for $|\lambda| > 1$, for large N and M, we get

$$\mathrm{var}[f_{N,M}(\lambda_1, \lambda_2)] = \int_{-1}^{1} W(x_1)W(x_2)W(x'_1)W(x'_2)C(x, x')dx_1 dx_2 dx'_1 dx'_2,$$

where

$$C(x,x') = \operatorname{cov}\left[\hat{I}_{N,M}\left(\lambda_1 - \frac{x_1}{M_N}, \lambda_2 - \frac{x_2}{L_M}\right), \hat{I}_{N,M}\left(\lambda_1 - \frac{x_1'}{M_N}, \lambda_2 - \frac{x_2'}{L_M}\right)\right].$$

We consider the three following subsets:

- $L_1 = \{(x_1, x_1') \in [-1,1]^2;\quad |x_1 - x_1'| > \sigma_N\}$,
- $L_2 = \{(x_2, x_2') \in [-1,1]^2;\quad |x_2 - x_2'| > \sigma_M'\}$,
- $L_3 = \{(x_1, x_1', x_2, x_2') \in [-1,1]^4;\quad |x_1 - x_1'| \leq \sigma_N \text{ or } |x_2 - x_2'| \leq \sigma_M'\}$,

where σ_N and σ_M' are two nonnegative real sequences, converging to 0. We split the integral into an integral over the subregion L_3 and an integral over $L_1 \times L_2$.

$$\operatorname{var}[f_{N,M}(\lambda_1, \lambda_2)] = \int_{L_3} + \int_{L_1 \times L_2} \triangleq J_1 + J_2.$$

By Cauchy Schwartz inequality and theorem 3.1, we obtain

$$J_1 \leq C\left[\int_{|x_2 - x_2'| \leq \sigma_M'} W(x_2)W(x_2')dx_2dx_2' + \int_{|x_1 - x_1'| \leq \sigma_N} W(x_1)W(x_1')dx_1dx_1'.\right]$$

where C is a constant. Thus we obtain

$$J_1 \leq C\,[\sup(W)]^2\,[\sigma_N + \sigma_M']. \tag{16}$$

It remains to show that J_2 converges to zero. For simplicity, we define

$$\lambda_{1,1} = \lambda_1 - \frac{x_1}{M_N};\quad \lambda_{1,2} = \lambda_1 - \frac{x_1'}{M_N};\quad \lambda_{2,1} = \lambda_2 - \frac{x_2}{L_M};\quad \lambda_{2,2} = \lambda_2 - \frac{x_2'}{L_M},$$

We first show that $C(x,x')$ converges to zero uniformly in $x_1, x_2, x_1', x_2' \in [-1,1]$. Indeed from the equalities (10) and (11) we have

$$\begin{aligned} Bais &\triangleq \mathbf{E}\left[\hat{I}_{N,M}(v_1,v_2)\right] - \hat{I}_{N,M}(v_1,v_2) \\ &= F_{p,\alpha}^{-1}[C_\alpha]^{-p/\alpha} \int_{-\infty}^{\infty} \frac{\mathcal{R}e\left(e^{iuI_{N,M}(v_1,v_2)}\right) - e^{-C_\alpha |u|^\alpha \phi_{N,M}(v_1,v_2)}}{|u|^{1+p}} du. \end{aligned}$$

Thus the expression of the covariance becomes

$$C(x,x') = F_{p,\alpha}^{-2}C_\alpha^{-\frac{2p}{\alpha}} \int_{\mathbb{R}^2} \mathbf{E}\left[\prod_{k=1}^{2} \cos\left(u_k I_{N,M}(\lambda_{1,k},\lambda_{2,k})\right)\right] - \exp\left\{-C_\alpha \sum_{k=1}^{2} |u_k|^\alpha \psi_{N,M}(\lambda_{1,k},\lambda_{2,k})\right\} \frac{du_1du_2}{|u_1u_2|^{1+p}}$$

The following equality $2\cos x \cos y = \cos(x+y) + \cos(x-y)$, implies that

$$\mathbf{E}\left[\prod_{k=1}^{2} \cos\left(u_k I_{N,M}(\lambda_{1,k},\lambda_{2,k})\right)\right] = \frac{1}{2}\exp\left[-C_\alpha \int |\tau_1\tau_2|\, A_{N,M}(v_1,v_2)d\mu(v_1,v_2)\right] + \frac{1}{2}\exp\left[-C_\alpha \int |\tau_1\tau_2|\, B_{N,M}(v_1,v_2)d\mu(v_1,v_2)\right]$$

Where

$$A_{N,M}(v_1,v_2) = \left|\sum_{k=1}^{2} u_k H_N(\tau_1\lambda_{1,k} - \tau_1 v_1)H_M(\tau_2\lambda_{2,k} - \tau_2 v_2)\right|^\alpha$$

$$B_{N,M}(v_1,v_2) = \left|\sum_{k=1}^{2} (-1)^{k-1} u_k H_N(\tau_1\lambda_{1,k} - \tau_1 v_1)H_M(\tau_2\lambda_{2,k} - \tau_2 v_2)\right|^\alpha .$$

By substituting in the expression for $C(x,x')$ and changing the variable u_2 to $(-u_2)$ in the second terme, we obtain

$$C(x,x') = F_{p,\alpha}^{-2}C_\alpha^{-\frac{2p}{\alpha}} \int_{\mathbb{R}^2} \left(e^{-K} - e^{-K'}\right) \frac{du_1du_2}{|u_1u_2|^{1+p}}, \tag{17}$$

where

$$K = C_\alpha \int_{\mathbb{R}^2} \left|(\tau_1\tau_2)^{\frac{1}{\alpha}} \sum_{k=1}^{2} u_k H_N(\tau_1\lambda_{1,k} - \tau_1 v_1)H_M(\tau_2\lambda_{2,k} - \tau_2 v_2)\right|^\alpha d\mu(v_1,v_2)$$

$$K' = C_\alpha \sum_{k=1}^{2} |u_k|^\alpha \int_{\mathbb{R}^2} |H_N(\tau_1\lambda_{1,k} - v_1)H_M(\tau_2\lambda_{2,k} - v_2)|^\alpha \phi\left(\frac{v_1}{\tau_1},\frac{v_2}{\tau_1}\right) dv_1dv_2$$

Since $K, K' > 0$, $\left|e^{-K} - e^{-K'}\right| \le |K - K'| \exp\left\{|K-K'| - K'\right\}$

Using the inequality (2) we obtain:

$|K - K'| \leq 2C_\alpha \tau_1\tau_2|u_1u_2|^{\frac{\alpha}{2}} Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2})$, where

$$Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2}) = \int_{-\Omega_1}^{\Omega_1}\int_{-\Omega_2}^{\Omega_2}\left|H_N(\tau_1\lambda_{1,1}-\tau_1u_1)H_M(\tau_2\lambda_{2,1}-\tau_2u_2)\right|^{\frac{\alpha}{2}}$$

$$\times\left|H_N(\tau_1\lambda_{1,2}-\tau_1u_1)H_M(\tau_2\lambda_{2,2}-\tau_2u_2)\right|^{\frac{\alpha}{2}}\phi(u_1,u_2)du_1du_2$$

Let show now that $Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2})$ converges to zero. Indeed, since ϕ is bounded on $[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]$, we have

$$Q_{N,M}(\lambda_{1,1};\lambda_{1,2};\lambda_{2,1};\lambda_{2,2}) \leq \int_{-\Omega_1}^{\Omega_1}\left|H_N(\tau_1\lambda_{1,1}-\tau_1u_1)H_N(\tau_1\lambda_{1,2}-\tau_1u_1)\right|^{\frac{\alpha}{2}}du_1$$

$$\times\sup(\phi)\int_{-\Omega_2}^{\Omega_2}\left|H_M(\tau_2\lambda_{2,1}-\tau_2u_2)H_M(\tau_2\lambda_{2,2}-\tau_2u_2)\right|^{\frac{\alpha}{2}}du_2 \qquad (18)$$

From the definition of H_N we write

$$\int_{-\Omega_1}^{\Omega_1}\left|H_N(\tau_1\lambda_{1,1}-\tau_1v_1)H_N(\tau_1\lambda_{1,2}-\tau_1v_1)\right|^{\frac{\alpha}{2}}dv_1 =$$

$$\int_{-\Omega_1}^{\Omega_1}\frac{1}{B'_{\alpha,N}}\left|\frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,1}-\tau_1v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,1}-\tau_1v_1)\right]}\right|^{k\alpha}\left|\frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,2}-\tau_1v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2}-\tau_1v_1)\right]}\right|^{k\alpha}dv_1.$$

a)First step: We show that the denominators of the first and second fraction under the last integral do not vanish for the same v_1, so we suppose there exist v_1 belonging to $[-\Omega_1,\Omega_1]$ and $z,z'\in\mathbb{Z}$ such that: $\tau_1\lambda_{1,1}-\tau_1v_1 = 2z\pi$ and $\tau_1\lambda_{1,2}-\tau_1v_1 = 2z'\pi$. Since $\lambda_{1,1}\neq\lambda_{1,2}$, then z and z' are different. Therefore $z-z' = \frac{\tau_1}{2\pi}(\lambda_{1,1}-\lambda_{1,2})$. Hence, $|z-z'| = \frac{1}{w_1}|\lambda_{1,1}-\lambda_{1,2}|$. As $\lim_{N\to\infty}|\lambda_{1,1}-\lambda_{1,2}| = 0$, consequently for a large N we get: $|z-z'| < \frac{1}{2}$. Thus, we obtain a contradiction with the fact that z and z' are different integers. *b) second step:* We assume there exist q points, $V_1,V_2,\cdots,V_q\in[-\Omega_1,\Omega_1]$ such that for $j=1,2,\cdots,q$ $\tau_1\lambda_{1,1}-\tau_1V_j\in 2\pi\mathbb{Z}$, therefore $\frac{\lambda_{1,1}}{w_1}-$

$\frac{V_j}{w_1} \in \mathbb{Z}$, and we assume there exist q' points $V'_1, V'_2, \cdots, V'_{q'} \in [-\Omega_1, \Omega_1]$ such that, for $i = 1, 2, \cdots, q'$ $\frac{\lambda_{1,2}}{w_1} - \frac{V'_i}{w_1} \in \mathbb{Z}$.

Showing that, $|V_j| \neq \Omega_1$, $|V'_i| \neq \Omega_1$ for $1 \leq j \leq q$ and $1 \leq i \leq q'$. Indeed, $-1 < \frac{\lambda_1 - \Omega_1}{w_1} < 0$ and $0 < \frac{\lambda_1 + \Omega_1}{w_1} < 1$ because $w_1 > 2\Omega_1$. Hence $\frac{\lambda_1 - \Omega_1}{w_1} \notin \mathbb{Z}$ and $\frac{\lambda_1 + \Omega_1}{w_1} \notin \mathbb{Z}$. On the other hand, $\frac{\lambda_1 - \frac{s_1}{M_N}}{w_1} - \frac{\Omega_1}{w_1} \to \frac{\lambda_1 - \Omega_1}{w_1}$ as $N \to \infty$. For a large N we get that $\left[\frac{\lambda_1 - \Omega_1}{w_1}\right]_E < \frac{\lambda_{1,1}}{w_1} - \frac{\Omega_1}{w_1} < 1 + \left[\frac{\lambda_1 - \Omega_1}{w_1}\right]_E$, where $[x]_E$ denote the integer part of x.Hence, $\frac{\lambda_{1,1}}{w_1} - \frac{\Omega_1}{w_1} \notin \mathbb{Z}$. In the same manner we show that $\frac{\lambda_{1,1}}{w_1} + \frac{\Omega_1}{w_1} \notin \mathbb{Z}$. Similarly it can be shown that: $\frac{\lambda_{1,2} + \Omega_1}{w_1} \notin \mathbb{Z}$. Thus $|V_j| \neq \Omega_1$ and $|V'_i| \neq \Omega_1$.

c) third step: We classify V_j and V'_i by increasing order:

$-\Omega_1 < V_{j_1} < V_{j_2} < \cdots < V_{j_{q+q'}} < \Omega_1$ and we write the integral in the following manner:

$$\begin{aligned} I &= \int_{-\Omega_1}^{\Omega_1} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]} \right|^{ka} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]} \right|^{ka} dv_1 \\ &= I_1 + \sum_{i=1}^{q+q'} I_{2,i} + \sum_{i=1}^{q+q'-1} I_{3,i} + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{-\Omega_1}^{V_{j_1} - \delta(N)} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]} \right|^{ka} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]} \right|^{ka} dv_1. \\ I_{2,i} &= \int_{V_{j_i} - \delta(N)}^{V_{j_i} + \delta(N)} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]} \right|^{ka} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]} \right|^{ka} dv_1. \\ I_{3,i} &= \int_{V_{j_i} + \delta(N)}^{V_{j_{i+1}} - \delta(N)} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]} \right|^{ka} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]} \right|^{ka} dv_1. \\ I_4 &= \int_{V_{j_{q+q'}} + \delta(N)}^{\Omega_1} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,1} - \tau_1 v_1)\right]} \right|^{ka} \left| \frac{\sin\left[\frac{n}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2} - \tau_1 v_1)\right]} \right|^{ka} dv_1 \end{aligned}$$

where $\delta(N)$ is a nonnegative real number converging to zero and satisfying the following inequalities :

$-\Omega_1 < V_{j_1} - \delta(N) < V_{j_1} + \delta(N) < V_{j_2} - \delta(N) < V_{j_2} + \delta(N) < \cdots < V_{j_{q+q'}} - \delta(N) < V_{j_{q+q'}} + \delta(N) < \Omega_1$ and $\delta(N) < \left|\frac{\lambda_{1,1}-\lambda_{1,2}}{2}\right|$.

First, we show that the first integral converges to zero. We know that for a large N we have $\lambda_{1,1} < \Omega_1$.

Since there is no v between$-\Omega_1$ and $V_{j_1} - \delta(N)$ on which the denominators are vanishing.

$$I_1 \leq \frac{V_{j_1} - \delta(N) + \Omega_1}{\inf\left[\left|\sin\frac{\tau_1\delta(N)}{2}\right|^{k\alpha}, \left|\sin\frac{\tau_1(\lambda_{1,1}+\Omega_1)}{2}\right|^{k\alpha}\right]} \frac{1}{\inf\left[\left|\sin\frac{\tau_1(\lambda_{1,2}-V_{j_1}+\delta(N))}{2}\right|^{k\alpha}, \left|\sin\frac{\tau_1(\lambda_{1,2}+\Omega_1)}{2}\right|^{k\alpha}\right]}.$$

By substituting for V_{j_1} in the last inequality, we obtain : $\left|\sin\frac{\tau_1(\lambda_{1,2}-V_{j_1}+\delta(N))}{2}\right|^{k\alpha} = \left|\sin\frac{\tau_1|\lambda_{1,2}-\lambda_{1,1}+\delta(N)|}{2}\right|^{k\alpha}$. For a large N we have $\frac{\tau_1|\lambda_{1,2}-\lambda_{1,1}+\delta(N)|}{2} \leq \tau_1\Omega_1 + \frac{\tau_1\delta(N)}{2} < \pi - \frac{\tau_1\delta(N)}{2}$. On the other hand two cases are possible :

1) if $\lambda_{1,2} - \lambda_{1,1} > 0$ we have $|\lambda_{1,2} - \lambda_{1,1} + \delta(N)| = \lambda_{1,2} - \lambda_{1,1} + \delta(N) > \delta(N)$

2) if $\lambda_{1,2} - \lambda_{1,1} < 0$, since $|\lambda_{1,2} - \lambda_{1,1}| > 2\delta(N)$, we have $|\lambda_{1,2} - \lambda_{1,1} + \delta(N)| = \lambda_{1,1} - \lambda_{1,2} - \delta(N) > \delta(N)$.

Therefore $\frac{\tau_1\delta(N)}{2} < \frac{\tau_1|\lambda_{1,2}-\lambda_{1,1}+\delta(N)|}{2} < \pi - \frac{\tau_1\delta(N)}{2}$. Thus we get

$$I_1 \leq \frac{V_{j_1} - \delta + \Omega_1}{\left|\sin\frac{\tau_1\delta(N)}{2}\right|^{2k\alpha}}.$$

For the integral $I_{2,i}$, we bound the first fraction under integral by $n^{k\alpha}$. $I_{2,i} \leq n^{k\alpha}\int_{V_{j_i}-\delta(N)}^{V_{j_i}+\delta(N)} \frac{1}{\left|\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2}-\tau_1 v_1)\right]\right|^{k\alpha}} dv_1$. By substituting for V_{j_i} in the last inequality and putting $v_1 = v - \frac{2k\pi}{\tau_1}$ we get $I_{2,i} \leq n^{k\alpha}\int_{\lambda_{1,1}-\delta(N)}^{\lambda_{1,1}+\delta(N)} \frac{1}{\left|\sin\left[\frac{1}{2}(\tau_1\lambda_{1,2}-\tau_1 v)\right]\right|^{k\alpha}} dv$. since $|\lambda_{1,1} - v| < \delta(N)$, it is easy to see that $|\lambda_{1,2} - v| \geq |\lambda_{1,2} - \lambda_{1,1}| + |\lambda_{1,1} - v| \geq |\lambda_{1,2} - \lambda_{1,1}| - \delta(N) > \frac{|\lambda_{1,2}-\lambda_{1,1}|}{2}$

Since $\delta(N)$ converges to zero, for a large N,we have $\delta(n) < \frac{2}{\tau_1}(\pi - \frac{\tau_1}{2}|\lambda_{1,2} - \lambda_{1,1}|)$. Therefore $0 < \tau_1\frac{|\lambda_{1,2}-\lambda_{1,1}|}{4} < \tau_1\frac{|\lambda_{1,2}-v|}{2} < \tau_1\frac{|\lambda_{1,2}-\lambda_{1,1}|+\delta(N)}{2} < \pi$.

Rate of convergence

From (21) we have: $J_2 = O\left(S_{N,M}(\lambda_1, \lambda_2)\right)$ where

$$S_{N,M}(\lambda_1, \lambda_2) = \int_{-1}^{1} W(x_1)W(x_1')W(x_2)W(x_2')Q_{N,M}(\lambda_{1,1}; \lambda_{1,2}; \lambda_{2,1}; \lambda_{2,2})dx_1 dx_1' dx_2 dx_2'.$$

The Fubini's theorem implies

$$S_{N,M}(\lambda_1,\lambda_2)=\int_{-\Omega_1}^{\Omega_1}\int_{-\Omega_2}^{\Omega_2}\phi(v_1,v_2)\left(\int_{-1}^{1}\jmath(x_1,x_2,x_1',x_2')\hbar(x_1,x_2,x_1',x_2')dx_1dx_1'dx_2dx_2'\right)dv_1dv_2$$

where $\jmath(x_1,x_2,x_1',x_2')=W(x_1)W(x_1')W(x_2)W(x_2')$ and

$$\begin{aligned}\hbar(x_1,x_2,x_1',x_2') &= |H_N(\tau_1\lambda_{1,1}-\tau_1v_1)H_M(\tau_2\lambda_{2,1}-\tau_2v_2)|^{\frac{a}{2}}\\ &\quad |H_N(\tau_1\lambda_{1,2}-\tau_1v_1)H_M(\tau_2\lambda_{2,2}-\tau_2v_2)|^{\frac{a}{2}}.\end{aligned}$$

Let $u_1=\dfrac{x_1}{M_N}$ and $u_2=\dfrac{x_2}{L_M}$, we get:

$$\begin{aligned}S_{N,M}(\lambda_1,\lambda_2) &= \int_{-\Omega_1}^{\Omega_1}\int_{-\Omega_2}^{\Omega_2}\phi(v_1,v_2)\left(\int_{\frac{-1}{M_N}}^{\frac{1}{M_N}}W_N(u_1)\,|H_N(\tau_1\lambda_1-\tau_1u_1-\tau_1v_1)|^{\frac{a}{2}}\,du_1\right)^2\\ &\quad\times\left(\int_{-\frac{1}{L_M}}^{\frac{1}{L_M}}W_M(u_2)\,|H_M(\tau_2\lambda_2-\tau_2u_2-\tau_2v_2)|^{\frac{a}{2}}\,du_2\right)^2 dv_1dv_2.\end{aligned}$$

By two changes of variables first $w_1=\lambda_1-v_1$; $w_2=\lambda_2-v_2$ secondly $t_1=w_1-u_1$; $t_2=w_2-u_2$, we get:

$$S_{N,M}(\lambda_1,\lambda_2)=\int_{\lambda_1-\Omega_1}^{\lambda_1+\Omega_1}\int_{\lambda_2-\Omega_2}^{\lambda_2+\Omega_2}\phi(\lambda_1-w_1,\lambda_2-w_2)\,[G_N(w_1)]^2\,[G_M(w_2)]^2\,dw_1dw_2$$

where $G_N(w_1)=\int_{w_1-\frac{1}{M_N}}^{w_1+\frac{1}{M_N}}W_N(w_1-t_1)\,|H_N(\tau_1t_1)|^{\frac{a}{2}}\,dt_1$

and $G_M(w_2)=\int_{w_2-\frac{1}{L_M}}^{w_2+\frac{1}{L_M}}W_M(w_2-t_2)\,|H_M(\tau_2t_2)|^{\frac{a}{2}}\,dt_2$.

Since $-\Omega_1<\lambda_1<\Omega_1$ and $-\Omega_2<\lambda_2<\Omega_2$, we have:

$$S_{N,M}(\lambda_1,\lambda_2)\le\int_{-2\Omega_1}^{+2\Omega_1}\int_{-2\Omega_2}^{+2\Omega_2}\phi(\lambda_1-w_1,\lambda_2-w_2)\,[G_N(w_1)]^2\,[G_M(w_2)]^2\,dw_1dw_2.$$

Putting $\tau_1t_1=u_1$ and $\tau_2t_2=u_2$, the expression of $G_N(w_1)$ and $G_M(w_2)$ become:

$$\begin{aligned}G_N(w_1) &= \frac{1}{\tau_1}\int_{\tau_1w_1-\frac{\tau_1}{M_N}}^{\tau_1w_1+\frac{\tau_1}{M_N}}W_N(w_1-\frac{u_1}{\tau_1})\,|H_N(u_1)|^{\frac{a}{2}}\,du_1\\ G_M(w_2) &= \frac{1}{\tau_2}\int_{\tau_2w_2-\frac{\tau_2}{L_M}}^{\tau_2w_2+\frac{\tau_2}{L_M}}W_M(w_2-\frac{u_2}{\tau_2})\,|H_M(u_2)|^{\frac{a}{2}}\,du_2\end{aligned}$$

For the definition of the kernel H_N and the fact that is 2π periodic, for a large N and a large M, we have the following inequalities

$$\begin{aligned} G_N(w_1) &\le \frac{6M_N \sup(W) n^{k\alpha}}{\tau_1 (B'_{\alpha,N})^{\frac{1}{2}}} \int_0^{\frac{\pi}{n}} dt_1 + \frac{6M_N \sup(W)}{\tau_1 (B'_{\alpha,N})^{\frac{1}{2}}} \int_{\frac{\pi}{n}}^{\pi} \left(\frac{\pi}{t_1}\right)^{k\alpha} dt_1 \\ G_M(w_2) &\le \frac{6L_M \sup(W) m^{k\alpha}}{(\tau_2 B'_{\alpha,M})^{\frac{1}{2}}} \int_0^{\frac{\pi}{n}} dt_2 + \frac{6L_M \sup(W)}{\tau_2 (B'_{\alpha,M})^{\frac{1}{2}}} \int_{\frac{\pi}{m}}^{\pi} \left(\frac{\pi}{t_2}\right)^{k\alpha} dt_2 \end{aligned}$$

It follows from the lemma (2.1) that, $G_N(w_1) = O\left(\frac{M_N}{n^{\frac{1}{2}}}\right)$ and $G_M(w_2) = O\left(\frac{L_M}{m^{\frac{1}{2}}}\right)$. Thus $J_2 = O\left(\frac{(M_N)^2}{n}\frac{(L_M)^2}{m}\right)$. From the rate of convergence of J_1 in (10). we obtain:

$$\text{var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\sigma_N + \sigma'_M + \frac{(M_N)^2}{n}\frac{(L_M)^2}{m}\right).$$

In order to give a simplified rate of convergence for the variance, we take $M_N = n^c$, $L_M = m^{c'}$ where $\frac{1}{2k^2\alpha^2} < c < \frac{1}{2}$ and $\frac{1}{2k^2\alpha^2} < c' < \frac{1}{2}$, and we choose $\sigma_N = n^{-(1-2c)}$,$\sigma'_M = m^{-(1-2c')}$ with $d = 1-2c$, and $d' = 1-2c'$. Hence $\text{Var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\frac{1}{n^{(1-2c)}} + \frac{1}{m^{(1'2c')}} + \frac{1}{n^{(1-2c)}m^{(1-2c')}}\right)$. Thus

$$\text{Var}[f_{N,M}(\lambda_1,\lambda_2)] = O\left(\frac{1}{n^{(1-2c)}} + \frac{1}{m^{(1-2c')}}\right).$$

It remains to choose β and β' in the definition of $\delta(N)$ and $\delta(M)$ such that the limits in (22) go to zero with the same rate. Therefore β and β' must satisfy the following constraints:

$$2k\alpha - 1 - 2k\alpha\beta > 0 \tag{23}$$
$$2k\alpha - 1 - 2k\alpha\beta = \beta + k\alpha - 1 - (1-c)k\alpha \tag{24}$$
$$2k\alpha - 1 - 2k\alpha\beta' > 0 \tag{25}$$
$$2k\alpha - 1 - 2k\alpha\beta' = \beta' + k\alpha - 1 - (1-c')k\alpha \tag{26}$$

The equalities (24) and (26) imply that

$$\beta = \frac{2k\alpha - ck\alpha}{1+2k\alpha} \quad \text{and} \quad \beta' = \frac{2k\alpha - ck\alpha}{1+2k\alpha}.$$

Since $\frac{1}{2k^2\alpha^2} < c, c' < \frac{1}{2}$, it is clear that these β and β' satisfies the constraints (23) and (25) and the result follows.

Theorem 4.3. *Let* $-\Omega_1 < \lambda_1 < \Omega_1$ *and* $-\Omega_2 < \lambda_2 < \Omega_2$, *such that* $\phi(\lambda_1, \lambda_2) > 0$, *then*

$$\mathbf{E}\left|f_{N,M}(\lambda_1, \lambda_2) - [\phi(\lambda_1, \lambda_2)]^{\frac{p}{\alpha}}\right|^2 = o(1).$$

If ϕ satisfies the hypothesis $\mathcal{H}$ with $\gamma < 2k\alpha - 1$ and $M_N = n^c$

Proof: We show easily that:

$$\begin{aligned}\mathbf{E}\left|f_{N,M}(\lambda_1, \lambda_2) - [\phi(\lambda_1, \lambda_2)]^{\frac{p}{\alpha}}\right|^2 &= \left(\mathbf{E}\left[f_{N,M}(\lambda_1, \lambda_2)\right] - [\phi(\lambda_1, \lambda_2)]^{\frac{p}{\alpha}}\right)^2 \\ &- \operatorname{Var} f_{N,M}(\lambda_1, \lambda_2).\end{aligned}$$

From theorems 4.1 and 4.2 we get the result.

Theorem 4.4. *Let* (λ_1, λ_2) *belong to* $]-\Omega_1, \Omega_1[\times]-\Omega_2, \Omega_2[$ *such that* $\phi(\lambda_1, \lambda_2) > 0$. *If* $\alpha > \frac{1}{k}$, *then* $[f_{N,M}(\lambda_1, \lambda_{1,2})]^{\frac{\alpha}{p}}$ *converges in probability to* $\phi(\lambda_1, \lambda_2)$.

Proof: Using the following inequality: $|y^q - x^q| \leq \frac{q}{2}|y - x|(y^{q-1} + x^{q-1})$, $x, y \in \mathbb{R}^+$ and $q > 2$, we obtain

$$\begin{aligned}&\left|[f_{N,M}(\lambda_1, \lambda_2)]^{\frac{\alpha}{p}} - \phi(\lambda_1, \lambda_2)\right| \leq \\ &\quad \frac{\alpha}{2p}\left|f_{N,M}(\lambda_1, \lambda_2) - [\phi(\lambda_1, \lambda_2)]^{\frac{p}{\alpha}}\right|\left(([f_{N,M}(\lambda_1, \lambda_2)]^{\frac{\alpha}{p}-1} - [\phi(\lambda_1, \lambda_2)]^{\frac{\alpha-p}{\alpha}}\right).\end{aligned}$$

Thus we show easily that $[f_{N,M}(\lambda_1, \lambda_2)]^{\frac{\alpha}{p}}$ converges in probability to $\phi(\lambda_1, \lambda_2)$.

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