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Discrete Marshall-Olkin Extended Burr Type XII Distribution: Properties and Estimation

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A B S T R A C T

A new distribution with two parameters named discrete Marshall-Olkin extended Burr Type XII (DMOEBXII) distribution is introduced using the survival of discretizing approach. Some of the statistical properties are discussed for the new distribution such as survival, hazard rate, alternative hazard rate functions, moments, quantiles, and order statistics. Maximum likelihood method is applied under Type II censored samples for estimating the unknown parameters, survival and hazard rate function of the proposed model. A simulation study is carried out to illustrate the theoretical results of the maximum likelihood estimation. Finally, the DMOEBXII distribution is used to fit the number of day intervals between 109 consecutive coal-mining tragedies in the United Kingdom from 1875 to 1951, as well as 48 students' mathematics degrees at the Indian Institute of Technology in Kampur.

Keywords:

Discrete Marshall-Olkin extended Burr Type XII distribution; Discrete lifetime models; Order statistics; Type II censored; Maximum likelihood estimation; Squared error loss function; A

1. **Introduction**

The count data sets emerge in various fields like the yearly number of destructive earthquakes, number of patients of a specific disease in a hospital ward, failure of machines, number of patients due to coronavirus, number of monthly traffic accidents, hourly bacterial growth, and so on. Various discrete probability models have been utilized to model these kinds of data sets. Poisson and negative binomial distributions are frequently for modeling count observations. On the other hand, in the advanced scientific eon, the data generated from different fields is getting complex day by day, however, existing discrete models do not provide an efficient fit. Discretization of continuous distribution can be applied by using different approaches (survival discretization-mixed-Poisson-infinite series). The most widely used technique is the survival discretization approach by Roy (2003, 2004) proposed the discrete normal distribution and Discrete Rayleigh distribution, respectively. Krishna and Pundir (2009) introduced discrete Burr and discrete Pareto distributions.

Gómez-Déniz and Calderín-Ojeda (2011) proposed the discrete Lindley distribution. **AL-Huniti and AL-Dayian (2012)** introduced discrete Burr Type III distribution. Furthermore, **Akdogan *et al.* (2014)** introduced Point estimation of parameter in discrete burr distribution. **Para (2014)** proposed discrete generalized Burr Type XII distribution. **Kinaci *et al.* (2016)** applied Bayesian estimation for discrete Chen distribution. **Para and Jan (2016)** introduced discrete three parameter Burr Type XII and discrete Lomax distributions. Also, **AL-Metwally and Ibrahim (2020)** proposed discrete alpha power inverse Lomax distribution with application of COVID-19 data. In addition, **AL-Metwally *et al.* (2020)** introduced some of the statistical properties are obtained for the discrete Marshall-Olkin generalized exponential distribution. **Eliwa *et al.* (2020)** proposed discrete analogue of odd Weibull-G family of distributions. **Freitas *et al.* (2021)** applied Bayesian approach to estimating the parameter of the Discrete bilal distribution with right-censored data. **Gillariose *et al.* (2021)** proposed some of the statistical properties are obtained for the new the discrete Weibull Marshall–Olkin family of distributions. **Hegazy *et al.* (2021)** investigated Bayesian approach, under two types of loss function; squared error and linear exponential loss functions, to estimate the parameters of the Bayesian estimation and prediction of discrete Gompertz distribution. **UL-Haq *et al.* (2021)** presented the discrete Type II half-logistic exponential distribution with applications to COVID-19 data. **AL-Ghamdi *et al.* (2022)** introduced the discrete power-Ailamujia distribution.

The rest of the paper is organized as follows: the *discrete Marshall-Olkin extended Burr Type XII* (DMOEBXII) distribution is introduced, and some statistical properties are given in Section 2. While, in Section 3, moments and *maximum likelihood* (ML) estimators are derived of the unknown parameters. The efficiency of the introduced estimation is assessed via simulation study and results are presented, in Section 4. Section 5 provides two real applications of the DMOEBXII distribution. Conclusion is presented in Section 6.

1. Discretizing a Continuous Distribution

The general approach of discretizing a continuous variable can be used to construct a discrete model by introducing a grouping on the time axis see **Roy (2003, 2004)**. If the **crv**X has the sf, $S(x) = P(X \geq x)$and times are grouped into unit intervals so that the **drv** of X denoted= $[X]$; which is the largest integer less than or equal to, will have the *probability mass function* (pmf)

$$\mathrm{P(x) = S(x) - S(x+1)} \quad , \mathrm{x} = 0,1,2,\ldots \tag{1}$$

The pmf of the **drv**, dX can be viewed as discrete concentration of pdf of X. So, given any continuous distribution it is possible to construct corresponding discrete distribution using (1).

One of the advantages of applying this approach of discretizing is that the **sf** for discrete distributions has the same functional form of the **sf** for the continuous distributions; as a result, many reliability characteristics and properties remain unchanged. Thus, discretization of a continuous lifetime model according to this approach is an interesting and simple approach to derive a discrete lifetime model corresponding to the continuous one.

2.1. Construction of discrete Marshall-Olkin extended Burr Type XII distribution

Al-Saiari *et al.* (2017) presented mathematical and statistical properties and limitations of *MOEBXII* distribution along with application to real lifetime data and provided graphical illustrations of the dimensions of *MOEBXII* distribution. Also, they estimated the parameters using *ML* and Bayesian method.

The pdf of DMOEBXII distribution is given by

$$g(x;\alpha,\beta)=\frac{2\alpha\beta x e^{-x^2}\left(1-e^{-x^2}\right)^{\beta-1}}{\left[\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}\right]^2},\qquad x>0\quad,\quad \alpha,\beta>0 \tag{2}$$

where are $\alpha\ and\ \beta$ shape parameters and should be positive.
The corresponding **cdf** and **sf** are, respectively, given by

$$G(x;\alpha,\beta)=\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}},\qquad x>0\quad,\quad \alpha,\beta>0 \tag{3}$$

and

$$S(x)=1-\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}},\qquad x>0\quad,\quad \alpha,\beta>0 \tag{4}$$

Using (1) dX can be viewed as the discrete analogue to the continuous NAPTE variable X, and is commonly said to follow DMOEBXII distribution with two parameters $\alpha\ and\ \beta$, denoted by DMOEBXII (α,β) distribution, where the corresponding pmf of dX can be written as

$$p(x)=\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}}-\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}},\qquad x=0,1,2,\dots \tag{5}$$

and the **cdf**, **sf** and **hrf** are as follows:

$$F(x) = 1 - S(x) + P(x) = \frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}}, \qquad x = 0,1,2,\ldots \tag{6}$$

$$S(x) = 1 - F(x) + P(x) = 1 - \frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}, \qquad x = 0,1,2,\ldots \tag{7}$$

and

$$h(x) = \frac{P(x)}{S(x)}$$
$$= \frac{\left[\left(1-e^{-(x+1)^2}\right)^{\beta}\right]\left[\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}\right] - \left[\left(1-e^{-x^2}\right)^{\beta}\right]\left[\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}\right]}{\left[\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}\right]\left\{\left[\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}\right] - \left[\left(1-e^{-x^2}\right)^{\beta}\right]\right\}},$$

$$x = 0,1,2,\ldots;\ \alpha,\beta > 0. \tag{8}$$

There are some problems associated with the definition of $h(x)$, three of the more notable ones are given below:

a. $h(x)$is not additive for series system.
b. The cumulative hrf, $H(x) = \sum h(x) \neq -\ln S(x)$.
c. $h(x) \leq 1$ and it has the interpretation of a probability. [For more details, see **Xie *et al.* (2002) and Lai (2013) and (2014)].**

Therefore, it was necessary to find an alternative definition that is consistent with its continuous counterpart. **Roy and Gupta (1992)** provide an excellent alternative definition of a discrete **hrf** denoted by $h_1(x)$:

$$h_1(x) = \ln\left[\frac{s(x)}{s(x+1)}\right] = \ln\left[\frac{\left[\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}-\left(1-e^{-x^2}\right)^{\beta}\right]\left[\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}\right]}{\left[\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}\right]\left[\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}-\left(1-e^{-(x+1)^2}\right)^{\beta}\right]}\right],$$

$$x = 0,1,2,\ldots;\ \alpha,\beta > 0. \tag{9}$$

There is a relationship between $h_1(x)$ and $h(x)$, given by:

$$h(x) = 1 - e^{-h_1(x)} \tag{10}$$

The two concepts $h(x)$ and $h_1(x)$ have the same monotonic property, i.e., $h_1(x)$ is increasing (decreasing) if and only if $h(x)$ is increasing (decreasing).

Plots of **pmf** and **hrf** of DMOEBXII distribution are presented, respectively, in Figures 1 to 2, for some selected values of the parameters.

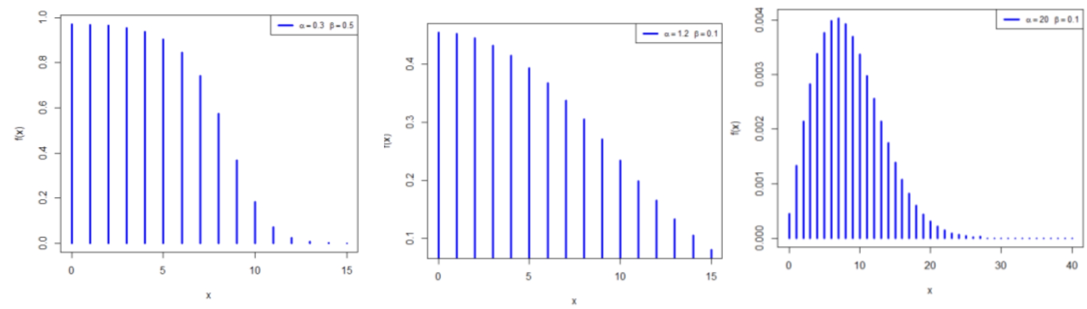


Figure 1: The plots of the probability mass function

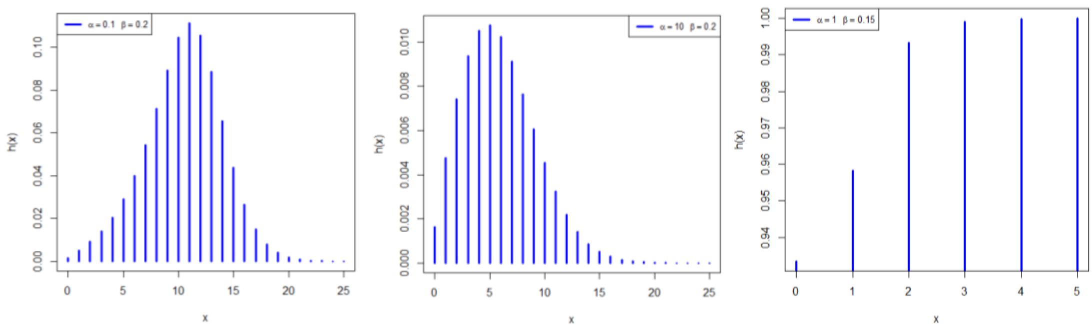


Figure 2: The plots of the hazard rate function

Figure 1, shows that the **pmf** of DMOEBXII distribution can be decreasing and right skewed according to the selected values of the parameters. Plots of pmf shows that the DMOEBXII distribution exhibits a long right tail compared with other commonly used distributions. Thus, it will affect long term reliability predictions, producing optimistic predictions of rare events occurring in the right tail of the distribution compared with other distributions. Figures 2 indicates that the hrf plots of DMOEBXII distribution are unimodal, increasing and right skewed shapes to the selected values of the parameters.

2.2 The main properties of discrete Marshall-Olkin extended Burr Type XII distribution

This section is devoted to obtain some important distributional properties of DMOEBXII (α, β) distribution, such as the mode, quantiles, r^{th} moments and order statistics.

2.2.1 Quantiles of discrete Marshall-Olkin extended Burr Type XII distribution

The u^{th} quantile x_u of a drv X, x_u, satisfies

$P(X \leq x_u) \geq u$ and $P(X \geq x_u) \geq 1-u$, i.e,$F(x_u - 1) < u \leq F(x_u)$. [For more details see **Rohatgi and Saleh (2001)**].

The u[th] quantile x_u of the DMOEBXII distribution (α, β) is given by:

$$x_u = \left\{\ln\left[1-\left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}} - 1, \qquad 0 < u < 1 \tag{11}$$

where [x] denotes the smallest integer greater than or equal to x and 0 < u < 1.

Proof

$P(X \leq x_u) \geq u$, from (6)

$$x_u \geq \left\{\ln\left[1-\left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}} - 1. \tag{12}$$

Similarly, if $P(X \geq x_u) \geq 1-u$, one obtains

$$x_u \leq \left\{\ln\left[1-\left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}}. \tag{13}$$

Combining (12) and (13), one gets,

$$\left\{\ln\left[1-\left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}} - 1 \leq x_u \leq \left\{\ln\left[1-\left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}}.$$

Hence, x_u is an integer value given by:

$$x_u = \left\{\ln\left[1-\left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}} - 1. \tag{14}$$

Thus, the median of DMOEBXII (α, β) distribution can be computed from (14) as follows

$$x_{0.5} = \left\{\ln\left[1-\left(\frac{0.5\alpha}{1-0.5(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}} - 1. \tag{15}$$

2.2.2 The moments of discrete Marshall-Olkin extended Burr Type XII distribution

a. The non-central moments of the discrete Marshall-Olkin extended Burr Type XII distribution

The non-central moments of DMOEBXII distribution can be obtained using (5) as follows:

$$\mu'_r = E(x^r) = \sum_{x=0}^{\infty} x^r \, p(x)$$

$$= \sum_{x=0}^{\infty} x^r \left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right], \qquad r = 1,2,3,4 \tag{16}$$

In particular, the mean (μ) of DMOEBXII distribution is given by

$$\mu = \sum_{x=0}^{\infty} x \left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right]. \tag{17}$$

b. The central moments of the discrete Marshall-Olkin extended Burr Type XII distribution

The *variance* (μ_2) of DMOEBXII distribution is

$$\mu_2 = \sum_{x=0}^{\infty} x^2 \left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right] - \left\{\sum_{x=0}^{\infty} x \left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right]\right\}^2. \tag{18}$$

In general, the central moments can be derived using the relation between the central and non-central moments as given below

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \mu'_{r-j}, \qquad r = 1,2,\ldots \tag{19}$$

c. The standard moments of the discrete Marshall-Olkin extended Burr Type XII distribution

The r^{th} standard moments can be obtained as follows:

$$\gamma_r = E\left(\frac{x-\mu}{\sigma}\right)^r. \tag{20}$$

The skewness and kurtosis of the DMOEBXII distribution are given by, respectively,

$\alpha_3 = \frac{\mu_3}{\mu_2^{1.5}}$ and $\alpha_4 = \frac{\mu_4}{\mu_2^2}$, Where μ_r, is given by (18) and r = 1, 2, ….

2.2.3 The order statistic of the discrete Marshall-Olkin extended Burr Type XII distribution

Let $F(x;\alpha,\beta)$; the cdf of the i^{th} order statistic for a random sample $X_1, X_2, \ldots, X_n$, from the DMOEBXII (α,β), is given by

$$F_i(x;\, \alpha, \beta) = \sum_{r=i}^{n} \binom{n}{r} \left[F(x;\, \beta, \alpha)\right]^r \left[1 - F(x;\, \beta, \alpha)\right]^{n-r}. \tag{21}$$

Using the binomial expansion for $[1 - F_i(x; \alpha, \beta)]^{n-r}$ and substituting (6) in (21), where

$$F_i(x;\ \alpha,\beta) = \sum_{r=i}^{n} \binom{n}{r} [F(x;\ \alpha,\beta)]^r \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F(x;\ \alpha,\beta)]^j$$

$$= \sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\frac{\left(1 - e^{-(x+1)^2}\right)^{\beta}}{\alpha + (1-\alpha)\left(1 - e^{-(x+1)^2}\right)^{\beta}} \right]^{r+j}. \tag{22}$$

Special cases

Case I: If i=1 in (22) one can obtain the distribution function of the first order statistic, as given below

$$F_1(x;\ \alpha,\beta) = 1 - [1 - F(x;\ \alpha,\beta)]^n = 1 - \left[1 - \left(\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}} \right) \right]^n. \tag{23}$$

Case II: If i = n in (22) the distribution function of the largest order statistic, as follows:

$$F_n(x;\ \alpha,\beta) = [F(x;\ \alpha,\beta)]^n = \left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}} \right]^n. \tag{24}$$

Suppose that $X_1, X_2, X_3, ..., X_n$ is a random sample from the DMOEBXII distribution with two parameters α and β. Let $X_{1:n}, X_{2:n}, X_{3:n}, ... , X_{n:n}$ denote the corresponding order statistics. Then, the pmf of $X_{i:n}$, is defined by:

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!} \int_{F(x-1)}^{F(x)} v^{i-1}(1-v)^{n-i} dv. \tag{25}$$

Using the binomial expansion for $(1 - v)^{n-i}$, then the pmf in (26).

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \int_{F(x-1)}^{F(x)} v^{i+v-1} dv$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left(\frac{1}{i+j}\right)$$

$$\times \left[\left[\frac{\left(1 - e^{-(x+1)^2}\right)^{\beta}}{\alpha + (1-\alpha)\left(1 - e^{-(x+1)^2}\right)^{\beta}} \right]^{i+j} - \left[\frac{\left(1 - e^{-x^2}\right)^{\beta}}{\alpha + (1-\alpha)\left(1 - e^{-x^2}\right)^{\beta}} \right]^{i+j} \right]. \tag{26}$$

The pmf of the smallest order statistic is obtained by substituting i=1 in (26) as follows:

$$P(X_{1:n}=x)=$$

$$n\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{j}\left(\frac{1}{1+j}\right)\times\left[\left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}}\right]^{1+j}-\left[\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right]^{1+j}\right]. \quad (27)$$

And, the pmf of largest order statistic is obtained by substituting $i=n$ in (26) as follows:

$$P(X_{n:n}=x)=\left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}}\right]^{n}-\left[\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right]^{n}. \quad (28)$$

Also, (22) can be used to obtain the pmf of the DMOEBXII (α,β) distribution, **(see Arnold *et al.* (2008)).**

3. Estimation of the Parameters of Discrete Marshall-Olkin Dxtended Burr Type XII Distribution

In this section, methods of moments and ML are used to derive the estimators of the parameters for the DMOEBXII distribution.

3.1 Method of moments

In this subsection, method of moments is applied to estimate the unknown parameters of the DMOEBXII distribution. The method of moments is based on equating the population moments, which are functions of the parameters to the corresponding sample moments and subsequently solving the two equations simultaneously. The first the second population and sample moments, respectively, are

$$\mu(\alpha,\beta)=\mu=\sum_{x=0}^{\infty}x\left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}}-\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right]. \quad (29)$$

$$\mu_2(\alpha,\beta)=\mu_2=\sum_{x=0}^{\infty}x^2\left[\frac{\left(1-e^{-(x+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x+1)^2}\right)^{\beta}}-\frac{\left(1-e^{-x^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x^2}\right)^{\beta}}\right]. \quad (30)$$

$$M_1=\frac{1}{n}\sum_{i=1}^{n}x_i \text{ and } M_2=\frac{1}{n}\sum_{i=1}^{n}x^2. \quad (31)$$

Then equating $\mu(\tilde{\alpha},\tilde{\beta})=M_1$ and $\mu_2(\tilde{\alpha},\tilde{\beta})=M_2$, (32) Where $\tilde{\alpha}\ and\ \tilde{\beta}$ are the estimators of $\alpha\ and\ \beta$

Since the moments of DMOEBXII distribution cannot be obtained in closed forms and (32) cannot be solved via ordinary techniques, therefore the estimates can be obtained numerically.

3.2 Method of maximum likelihood

In this section, method of ML is used to derive the estimators of the parameters for the DMOEBXII distribution

The method of ML is used to estimate the vector of two parameters, $\underline{\varphi} = (\alpha, \lambda)$ sf and hrf, of the DMOEBXII (α, β) distribution. Based on Type II censored samples, also confidence interval of the parameters (α, β) sf, and hrf are derived. Suppose that $X_1, X_2, ..., X_r$ is a Type II censored sample of size r obtained from a life test on n items whose lifetimes have a DMOEBXII (α, β) distribution. Then the likelihood function is

$$L\left(\underline{\varphi}, \underline{x}\right) \propto \{\prod_{i=1}^{r} p(x_i)\}[S(x_r)]^{n-r}, \tag{33}$$

where p(x) and S(x) are given, respectively, by (5) and (7). The $X_{(i)}$'s are ordered times for $i = 1, 2, \dots\ r$

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left[\prod_{i=1}^{r} \frac{\left(1-e^{-(x_i+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}\right] \left[1 - \frac{\left(1-e^{-x_r^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}}\right]^{n-r}. \tag{34}$$

The natural logarithm of the likelihood function is given by

$$\ell = lnL\left(\underline{\varphi}; \underline{x}\right) \propto ln\prod_{i=1}^{r}\left[\frac{\left(1-e^{-(x_i+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}\right]$$

$$+(n-r)\ln\left[1-\frac{\left(1-e^{-x_r^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}}\right]. \tag{35}$$

$$= \sum_{i=1}^{r} ln\left[\frac{\left(1-e^{-(x_i+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}} - \frac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}\right] + (n-r)\ln\left[1-\frac{\left(1-e^{-x_r^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}}\right]. \tag{36}$$

Considering the two parameters, α and β are unknown and differentiating the log likelihood function in (36), with respect to α and β, one obtains

$$\frac{\partial \ell}{\partial \alpha}=\sum_{i=1}^{r}\frac{\dfrac{-\left(1-e^{-(x_i+1)^2}\right)^{\beta}\left[1-\left(1-e^{-(x_i+1)^2}\right)^{\beta}\right]}{\left[\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}\right]^2}+\dfrac{\left(1-e^{-x_i^2}\right)^{\beta}\left[1-\left(1-e^{-x_i^2}\right)^{\beta}\right]}{\left[\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}\right]^2}}{\dfrac{\left(1-e^{-(x_i+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}}-\dfrac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}}$$

$$-(n-r)\frac{\left(1-e^{-x_r^2}\right)^{\beta}\left[1-\left(1-e^{-x_r^2}\right)^{\beta}\right]}{\left[\alpha+(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}\right]\left\{\left[\alpha+(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}\right]-\left(1-e^{-x_r^2}\right)^{\beta}\right\}} \tag{37}$$

and

$$\frac{\partial \ell}{\partial \beta}=$$

$$\sum_{i=1}^{r}\left\{\frac{\left[\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}\right]\left[\left(1-e^{-(x_i+1)^2}\right)^{\beta}\ln\left(1-e^{-(x_i+1)^2}\right)\right]-\left[\left(1-e^{-(x_i+1)^2}\right)^{\beta}\right]\left[(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}\ln\left(1-e^{-(x_i+1)^2}\right)\right]}{\left[\dfrac{\left(1-e^{-(x_i+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}}-\dfrac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}\right]\left[\left[\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}\right]^2\right]}-\right.$$

$$\left.\frac{\left[\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}\right]\left[\left(1-e^{-x_i^2}\right)^{\beta}\ln\left(1-e^{-x_i^2}\right)\right]-\left[\left(1-e^{-x_i^2}\right)^{\beta}\right]\left[(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}\ln\left(1-e^{-x_i^2}\right)\right]}{\left[\dfrac{\left(1-e^{-(x_i+1)^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-(x_i+1)^2}\right)^{\beta}}-\dfrac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}\right]\left[\left[\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}\right]^2\right]}\right\}-(n-$$

$$r)\left[\frac{\left[\alpha+(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}\right]\left[\left(1-e^{-x_r^2}\right)^{\beta}\ln\left(1-e^{-x_r^2}\right)\right]-\left[\left(1-e^{-x_r^2}\right)^{\beta}\right]\left[(1-\alpha)\left(1-e^{-x_r^2}\right)^{\beta}\ln\left(1-e^{-x_r^2}\right)\right]}{\left[1-\dfrac{\left(1-e^{-x_i^2}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}}\right]\left[\left[\alpha+(1-\alpha)\left(1-e^{-x_i^2}\right)^{\beta}\right]^2\right]}\right]. \tag{38}$$

Then the ML estimators of the parameters, denoted by $\hat{\alpha}$ and $\hat{\beta}$ are derived by equating the two nonlinear likelihood (37) and (38) to zeros and solving numerically.

Depending on the in-variance property, the ML estimators of S(x), h(x) and h_1(x) can be obtained by replacing α and β with their corresponding ML estimators $\hat{\alpha}$ and $\hat{\beta}$, respectively, in (7), (8) and (9) as given below

$$\hat{S}_{ML}(x)=1-\frac{\left(1-e^{-x^2}\right)^{\hat{\beta}}}{\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-x^2}\right)^{\hat{\beta}}},\qquad x=0,1,2,\ldots \tag{39}$$

$$\hat{h}_{ML}(x)=\frac{\left[\left(1-e^{-(x+1)^2}\right)^{\hat{\beta}}\right]\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-x^2}\right)^{\hat{\beta}}\right]-\left[\left(1-e^{-x^2}\right)^{\hat{\beta}}\right]\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-(x+1)^2}\right)^{\hat{\beta}}\right]}{\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-(x+1)^2}\right)^{\hat{\beta}}\right]\left\{\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-x^2}\right)^{\hat{\beta}}\right]-\left[\left(1-e^{-x^2}\right)^{\hat{\beta}}\right]\right\}},\quad x=0,1,2,\ldots \tag{40}$$

$$\hat{h}_{1_{ML}}(x)=\ln\left[\frac{\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-x^2}\right)^{\hat{\beta}}-\left(1-e^{-x^2}\right)^{\hat{\beta}}\right]\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-(x+1)^2}\right)^{\hat{\beta}}\right]}{\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-x^2}\right)^{\hat{\beta}}\right]\left[\hat{\alpha}+(1-\hat{\alpha})\left(1-e^{-(x+1)^2}\right)^{\hat{\beta}}-\left(1-e^{-(x+1)^2}\right)^{\hat{\beta}}\right]}\right],\quad x=0,1,2,\ldots \tag{41}$$

When the sample size is large and the regularity conditions are satisfied, see **(Lehmann and Casella (1998)),** the asymptotic distribution of the ML estimators is $\underline{\varphi}$ ~ bivariate normal $\left(\underline{\varphi}, \mathrm{I}^{-1}\underline{x}\left(\underline{\varphi}\right)\right)$, where $\underline{\varphi}=(\alpha,\beta)$, $\hat{\varphi}=\left(\hat{\alpha},\hat{\beta}\right)$, and $I^{-1}(\varphi)$.

The asymptotic variance-covariance matrix of the ML estimators α and β, which is the inverse of the observed Fisher information matrix. The asymptotic observed Fisher information matrix can be obtained as follows:

$$I_{\underline{x}}\left(\underline{\varphi}\right)\approx\begin{bmatrix}-\left(\frac{\partial^2\ell}{\partial\alpha^2}\right) & -\left(\frac{\partial^2\ell}{\partial\alpha\,\partial\lambda}\right)\\ -\left(\frac{\partial^2\ell}{\partial\alpha\,\partial\lambda}\right) & -\left(\frac{\partial^2\ell}{\partial\lambda^2}\right)\end{bmatrix}_{(\hat{\alpha},\hat{\beta})}. \tag{42}$$

The asymptotic $100(1-\alpha)$ confidence interval for α, λ, $S_{ML}(x)$, $h_{ML}(x)$ and $h_{1_{ML}}(x)$ are given, respectively by:

$$L=\hat{\omega}-Z_{\frac{\alpha}{2}}\sigma_{\hat{\omega}}\quad and \qquad U=\hat{\omega}-Z_{\frac{\alpha}{2}}\sigma_{\hat{\omega}}, \tag{43}$$

where L and U are the lower and upper bound $\hat{\omega}$ is $\hat{\alpha}$, $\hat{\lambda}$, $\hat{S}(x)$, $\hat{h}(x)$ or $\hat{h}_1(x)$, Z is the $100(1-\frac{\alpha}{2})\%$ the standard normal percentile, $(1-\alpha)\%$ is the confidence coefficient, $\sigma_{\hat{w}}$ is the standard deviation and length $=\mathrm{U}-\mathrm{L}$.

4. Numerical Results

This section aims to investigate the precision of the theoretical results based on simulated and real data, by evaluating *relative absolute biases* (RABs) and *relative errors* (REs).

4.1 Simulation study

In this subsection, a simulation study is presented to illustrate the application of the various theoretical results developed in the previous section on the basis of generated

Data from DMOEBXII (α, β) distribution, for different sample sizes (n=30, 60 and 100) and using number of replications N=1000. The computations are performed using R package. The numerical procedures are performed according to the following algorithm.

Step 1: a random sample $X_1, X_2, \dots, X_n$ of sizes (n = 30, 60,100) these random samples are generated from DMOEBXII distribution using the following transformation:

$x_i = \left\{\ln\left[1 - \left(\frac{u\alpha}{1-u(1-\alpha)}\right)^{\frac{1}{\beta}}\right]^{-1}\right\}^{\frac{1}{2}} - 1, i = 1,2, \dots, n$ and u_i are random sample from uniform $(0,1)$ and then taking the ceiling.

Step 2: two different set values of the parameters are selected as,

Set 1$(\alpha = 1.5, \beta = 15)$ and Set 2 $\alpha = 50, \beta = 2$**.**

Step 3: For each model parameters and for each sample size, the ML and moments estimators are computed.

Step 4: Steps from 1 to 3 are repeated 1000 times for each sample size and for selected sets of the parameters. Then the averages, RABs, REs and variances of the estimates of the unknown parameters are computed.

The results of the simulation study are given in Tables 2 and 3. The average, RABs, REs and estimate risk of ML estimates of the parameters, sf and hrf are computed as follows:

1) Average $= \frac{\sum_{i=1}^{N} \hat{\varphi}_i}{N}$,
2) RAB $(\hat{\varphi}) = \frac{|bias(\hat{\varphi})|}{true\ value}$,
3) Relative error $(\hat{\varphi}) = \frac{RE(\hat{\varphi})}{true\ value}$,
4) Estimated risk $(\hat{\varphi}) = \frac{\sum_{i=1}^{N}(\hat{\varphi}-\varphi)^2}{N}$.

Table 2 shows the averages, RABs, Res for the parameters, sf and hrf estimates, also 95% confidence intervals where the initial values for the parameters are α=1.5, β=15 under three levels of $\frac{r}{n} \times 100$ percentage of uncensored observations Type II censoring 80% and 100%. Table 3 displays the same computational results, but for different initial values of the parameters α=50, β=2, at the same mission time t_0 from the DMOEBXII distribution for different sample sizes where (n=30, 60 and 100) and also level of Type II censoring 80% and 100% and number of replications, N = 1000.

Tables 1 and 2 show that the RABs and REs of the ML estimates of the parameters, sf and hrf, decrease as the sample size n increases, as expected. Furthermore, as the level of censoring decreases, so do the RABs and REs of the ML estimates of the parameters, sf and hrf estimates. As expected, the lengths of the confidence intervals decrease as sample size increases. These results are expected because decreasing the level of censoring means that the sample provides more information, increasing the accuracy of the estimates. In general, when r=n, all of the Type II censored sample results reduce to those of the entire sample.

Table 1: RABs, REs of ML estimates, 95% confidence intervals of the parameters, survival and hazard rate functions from DMOEBXII distribution for different sample sizes n, censoring level r and the replications N= 1000, $\alpha = 1.5$, $\beta = 15$, t_0=0.9

n	r	parameters	estimate	RABs	REs	LL	UL	length
30	24	α	1.16572	0.22285	0.38545	0.00938	2.32206	2.31268
		β	16.95243	0.13016	0.09315	14.15783	19.74697	5.58913
		R(t_0)	0.99956	0.00817	0.00904	0.98188	1.01803	0.03615
		h(t_0)	0.59667	0.02991	0.22721	0.33341	0.85995	0.52654
	30	α	1.13286	0.24476	0.40394	0	2.34469	2.34469
		β	16.83467	0.12231	0.09031	14.12567	19.54367	5.41800
		R(t_0)	0.99996	0.00779	0.00883	0.98229	1.01761	0.96529
		h(t_0)	0.60476	0.04385	0.27513	0.28596	0.92355	0.63758
50	40	α	1.24329	0.17114	0.33777	0.29948	2.25663	2.02668
		β	16.73467	0.11564	0.08781	14.10053	19.36881	5.26828
		R(t_0)	0.99995	0.00769	0.00878	0.98241	1.01749	0.03509
		h(t_0)	0.58694	0.01308	0.15029	0.14128	0.76108	0.34829
	50	α	1.38729	0.07513	0.22381	0.71585	2.05873	1.34289
		β	16.49483	0.09965	0.08151	14.04957	18.9401	4.89053
		R(t_0)	0.99995	0.00721	0.00849	0.98296	1.01693	0.03397
		h(t_0)	0.56763	0.02022	0.18682	0.35117	0.78409	0.43293
100	80	α	1.48269	0.01154	0.08771	1.21956	1.74582	0.52625
		β	15.20648	0.01376	0.03029	14.29768	16.11528	1.81761
		R(t_0)	0.99988	0.00139	0.00374	0.99241	1.00736	0.01494
		h(t_0)	0.57778	0.00269	0.06823	0.49873	0.65684	0.15811
	100	α	1.50109	0.00072	0.02197	1.43518	1.56699	0.13182
		β	15.01196	0.00079	0.00729	14.79326	15.23065	0.43739
		R(t_0)	0.99987	0.00093	0.00096	0.99795	1.00181	0.00385
		h(t_0)	0.57895	0.00068	0.03438	0.53911	0.61879	0.07968

Table 2: RABs, REs of ML estimates, 95% confidence intervals of the parameters, survival and hazard rate functions from DNAPTE distribution for different sample sizes n, censoring level r and the replications N= 1000, $\alpha = 50$, $\beta = 2$, t_0=0.9

n	r	parameters	estimates	RABs	REs	LL	UL	length
30	24	α	49.29487	0.01410	0.01679	47.61543	50.97431	3.35888
		β	2.64827	0.32413	0.40258	1.03797	4.25857	3.22061
		R(t_0)	0.95228	0.04102	0.21177	0.56484	1.33972	0.77487
		h(t_0)	0.24359	0.20246	0.81417	0	0.74094	0.74094
	30	α	49.38692	0.01226	0.01566	47.82093	50.95291	3.13198
		β	2.42765	0.21383	0.32697	1.11975	3.73555	2.6158
		R(t_0)	0.94205	0.02984	0.18063	0.61159	1.27252	0.66092
		h(t_0)	0.26185	0.14268	0.68348	0	0.67937	0.83503
50	40	α	49.67528	0.00649	0.01139	48.5356	50.81496	2.27937
		β	2.28342	0.14171	0.26618	1.21868	3.34815	2.12947
		R(t_0)	0.93404	0.02109	0.15183	0.65627	1.21181	0.55554
		h(t_0)	0.27542	0.09826	0.56721	0	0.6219	0.6219
	50	α	49.82491	0.00351	0.00837	48.98803	50.66179	1.67377
		β	2.30443	0.15221	0.27587	1.20093	3.40793	2.20700
		R(t_0)	0.93543	0.02261	0.1572	0.64783	1.22304	0.57521
		h(t_0)	0.27272	0.10709	0.59214	0	0.63444	0.63444
100	80	α	49.92764	0.00145	0.00538	49.38964	50.46564	1.07599
		β	2.15563	0.07781	0.19725	1.36663	2.94462	1.57798
		R(t_0)	0.92616	0.01247	0.11677	0.71252	1.13981	0.42728
		h(t_0)	0.28763	0.05828	0.43683	0.02786	0.55448	0.52662
	100	α	49.96835	0.00063	0.00356	49.61254	50.32416	0.71162
		β	2.09375	0.04688	0.15309	1.48137	2.70613	1.22475
		R(t_0)	0.92187	0.00777	0.08433	0.7532	1.09053	0.33733
		h(t_0)	0.29438	0.03619	0.3442	0.08117	0.50464	0.42053

5. Application

The main aim of this subsection is to demonstrate how the proposed methods can be used in practice. A real lifetime of two data sets is analyzed to illustrate the theoretical results.

Data Set I

Table 5 refers to an uncensored data set released by **Maguire *et al.* (1952)** that corresponds to day intervals between 109 consecutive coal-mining tragedies in the United Kingdom from 1875 to 1951. The sorted data is listed below:

Table 3: Data set of day intervals between 109 consecutive coal-mining tragedies in the United Kingdom from 1875 to 1951

1	4	4	7	11	13	15	15	17	18	19	19	20	20	22
23	28	29	31	32	36	37	47	48	49	50	54	54	55	59
59	61	61	66	72	72	75	78	78	81	93	96	99	108	113
114	120	120	120	123	124	129	131	137	145	151	156	171	176	182
188	189	195	203	208	215	217	217	224	228	233	255	271	275	275
275	286	291	312	312	312	315	326	326	329	330	336	338	345	348
354	361	364	369	378	390	457	467	498	517	566	644	745	871	1312
1357	1613	1630												

Table 4: The descriptive measures of day intervals between 109 consecutive coal-mining tragedies in the United Kingdom from 1875 to 1951

Min	Mean	SD	skewness	variance	kurtosis	Max
1	233.32	296.434	2.999	87873.331	10.526	1630

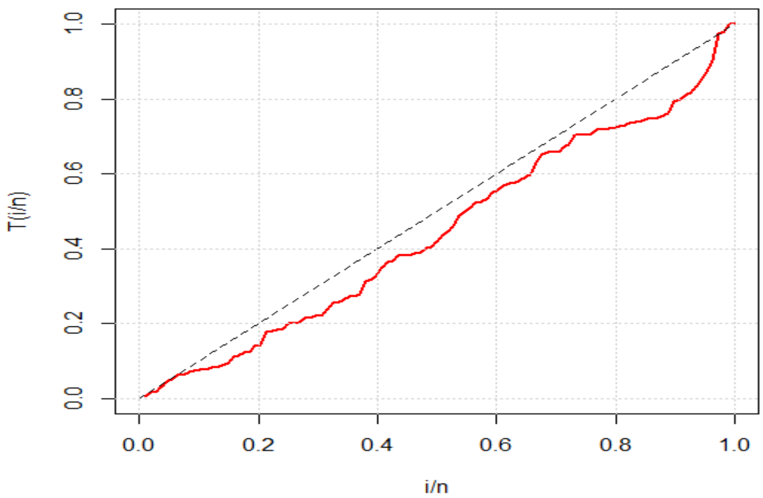


Figure 3. The TTT plot of the DMOEBXII model of day intervals between 109 consecutive coal-mining tragedies

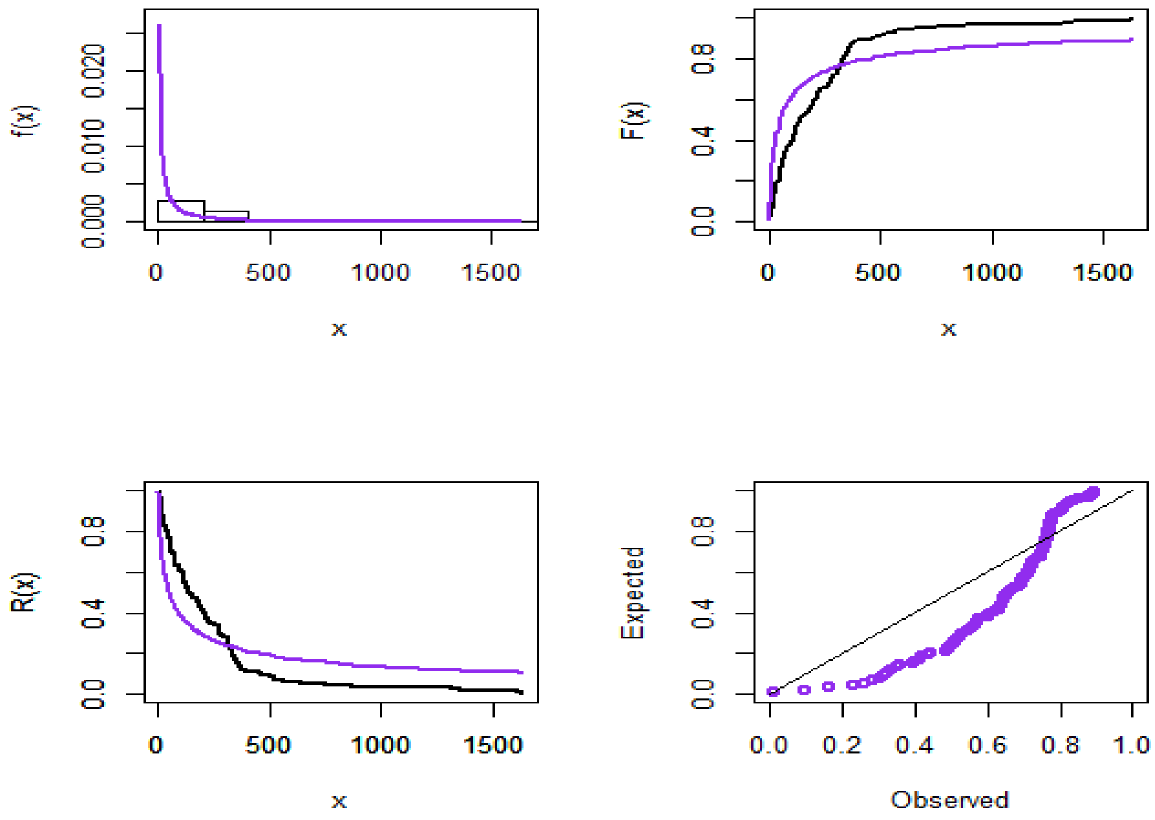


Figure 4: The Histogram, pdf, empirical cdf, empirical sf and the P-P plots of day intervals between 109 consecutive coal-mining tragedies

Data Set II

Table 5 shows the mathematics degrees of 48 students at the Indian Institute of Technology in Kampur. The data set was provided by **Gupta and Kundu (2009).**

Table 5: Data set of the mathematics degrees of 48 students at the Indian Institute of Technology in Kampur.

	29	25	50	15	13	27	15	18	7	7	8	19	12	18
5	21	15	86	21	15	14	39	15	14	70	44	6	21	58
19	50	23	11	6	34	18	28	34	12	37	4	60	20	23
40	65	19	31											

Table 6: The descriptive measures of the mathematics degrees of 48 students at the Indian Institute of Technology in Kampur

Min	First quantile	Median	mean	skewness	SD	Third quantile	variance	kurtosis	Max
4	14	19.5	25.9	1.3317	18.60478	34	346.1379	4.323312	86

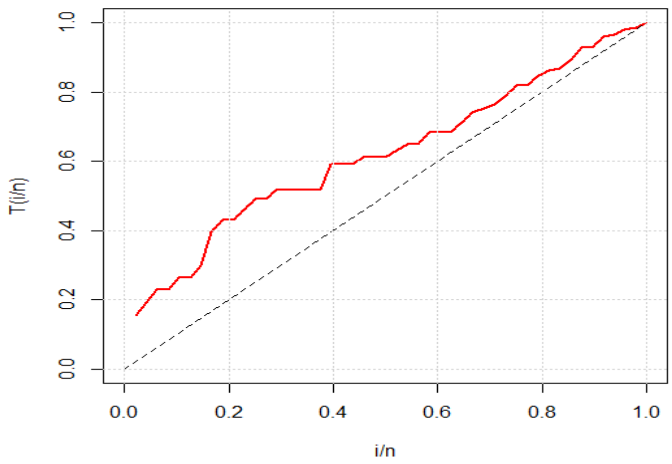
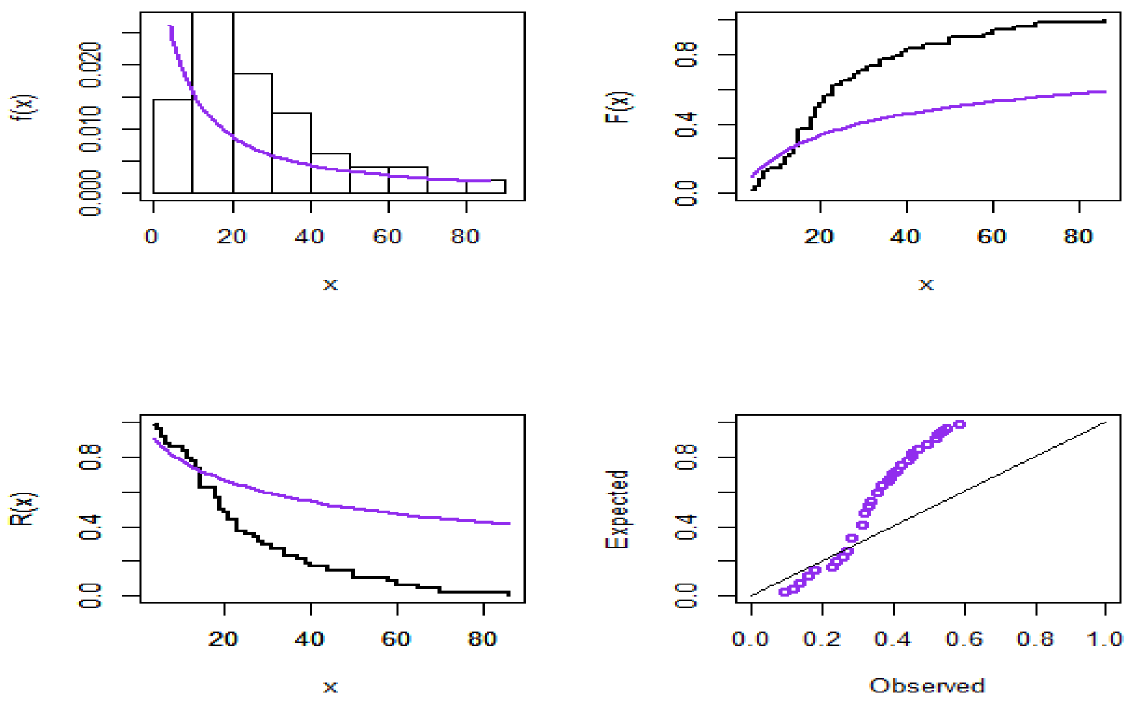


Figure 5. The TTT plot of the DMOEBXII model of the mathematics degrees of 48 students



at the Indian Institute of Technology in Kampur

Figure 6: The Histogram, pdf, empirical cdf, empirical sf and the P-P plots of the mathematics degrees of 48 students at the Indian Institute of Technology in Kampur

6. Conclusion

The Marshall-Olkin extended Burr Type XII distribution is introduced in this paper as a new discrete distribution. After constructed it, the parameter's probabilistic properties are investigated, and its parameters are estimated by ML and moments methods. To validate the theoretical results, comprehensive simulation results are obtained. The utility of the discrete Marshall-Olkin extended Burr Type XII distribution is demonstrated empirically through two applications data set of day intervals between 109 consecutive coal-mining tragedies in the United Kingdom from 1875 to 1951 and the mathematics degrees of 48 students at the Indian Institute of Technology in Kampur.

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