The Distribution of Sum, Product and Ratio for the Absolutely Continuous Bivariate Generalized Exponential Random Variables

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Abstract

Block and Basu bivariate exponential distribution is a standout amongst the most absolutely continuous bivariate distributions. This idea can be extended to the generalized exponential distribution also. In this case this distribution is called as the Block and Basu bivariate generalized exponential (BBBGE) distribution. Some properties of BBBGE distribution can be obtained as the moment generating function, median and mode. The exact forms for the distribution of sum; ratio, product of dependent variables follow the Block and Basu bivariate generalized exponential distribution are derived. The maximum likelihood estimation (MLE) procedure is performed for the parameters of the BBBGE distribution. A numerical illustration performed to see the performances of the MLEs.

Keywords: Block and Basu bivariate exponential distribution, moment generating function, Marshall-Olkin bivariate exponential distribution.

1 Introduction

Block and Basu bivariate generalized exponential (BBBGE) distribution has been obtained from the Marshal-Olkin bivariate generalized exponential (MOBGE) distribution by removing the singular part and that makes BBBGE distribution as an absolutely continuous bivariate distribution.

An important operation in probability theory is to obtain the distribution of the sum of two correlated random variables \( X_1 \) and \( X_2 \). Applications of the sums appear in many areas of mathematics, probability theory, physics and engineering. In many applications a random variable \( Z \) is a functionally related to two or more different random variables \( X_1 \) and \( X_2 \). A good example is the random signal \( S \) at the input of an amplifier consists of a random signal \( X_1 \) to which is added independent random
noise $X_2$. Hence the random signals $S$ is the sum of $X_1$ and $X_2$. Now an important question assess, what is the probability density function of the random variable $S$, which represents the amplifiers input. Also, many signal processing systems use electronic multipliers to multiply two signals together. If $X_1$ is the signal of one input and $X_2$ is another signal input, what is the probability density function of $P = X_1X_2$.

The ratios of two random variables $X_1$ and $X_2$ is the stress-strength model in the reliability theory. An important example is that model which describes the lifetimes of a component which has a random strength $X_1$ and subject to random stress $X_2$. These components fail at the time instant that the stress exceeds the strength and this component will function whenever $X_1 > X_2$. Hence the probability $P(X_2 > X_1) = P(2X_2 / (X_1 + X_2) < 1)$ is a measure of the reliability of the component.

The paper is organized as follows: In Section 2, the BBBGE distribution is introduced and the representations for the probability density function (pdf), cumulative distribution function (cdf), marginal distributions and moment generating function (mgf) are obtained. The exact forms for the distribution of sum; ratio and product of dependent variables follow the BBBGE distribution are derived in Section 3. The maximum likelihood estimation, estimated variance-covariance matrix and asymptotic confidence intervals for BBBGE distribution are provided in Section 4. Simulation results are presented in Section 5. Finally conclude the paper in Section 6.

2 The BBBGE Distribution

If $X$ has univariate generalized exponential (GE) distribution with the shape and scale parameters as $\alpha > 0$ and $\lambda > 0$ respectively, then the cdf and pdf of the GE distribution is as follows respectively,

$$F_{\text{GE}}(x; \alpha, \lambda) = (1 - e^{-\lambda x})^\alpha, \quad \gamma > 0$$

$$f_{\text{GE}}(x; \alpha, \lambda) = \alpha \lambda (1 - e^{-\lambda x})^{\alpha - 1} e^{-\lambda x}, \quad \gamma > 0, \alpha, \lambda > 0$$
Kundu and Gupta (2009) introduced that \((Y_1, Y_2)\) have MOBGE distribution if

The joint cdf of \((Y_1, Y_2)\) can be written as

\[
F_{MOBGE}(y_1, y_2) = \begin{cases} 
F_1(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
F_2(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
F_3(y) & \text{if } 0 < y_1 = y_2 = y < \infty 
\end{cases} \tag{2.1}
\]

Where

\[
F_1(y_1, y_2) = F_{GE}(y_1; \alpha_{13}) \cdot F_{GE}(y_2; \alpha_3) \\
F_2(y_1, y_2) = F_{GE}(y_1; \alpha_4) \cdot F_{GE}(y_2; \alpha_{23}) \\
F_3(y) = F_{GE}(y; \alpha_{123})
\]

Where \(\alpha_{13} = \alpha_1 + \alpha_3\), \(\alpha_{23} = \alpha_2 + \alpha_3\) and \(\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3\).

They observed that the joint distribution function of \((Y_1, Y_2)\) can be written as a mixture of an absolutely continuous part and a singular part as follows;

\[
F_{MOBGE}(y_1, y_2) = \frac{\alpha_{12}}{\alpha_{123}} F_a(y_1, y_2) + \frac{\alpha_3}{\alpha_{123}} F_s(y)
\]

where \(y = \min(y_1, y_2)\).

\(F_s(y) = (1 - e^{-y})^{\alpha_{23}}\),

and \(F_a(y_1, y_2) = \frac{\alpha_{12}}{\alpha_{123}} (1 - e^{-y_1})^{\alpha_1} (1 - e^{-y_2})^{\alpha_2} (1 - e^{-y})^{\alpha_3} - \frac{\alpha_3}{\alpha_{123}} (1 - e^{-y})^{\alpha_{23}}\).

Here \(F_s(\_\_\_)\) and \(F_a(\_\_\_)\) are the singular and the absolutely continuous part respectively.

The BBBGE distribution can be obtained from MOBGE distribution by removing the singular part and keeping only the continuous part. The joint pdf of BBBGE distribution can be written as

\[
f_{BBBGE}(x_1, x_2) = \begin{cases} 
f_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 < \infty \\
f_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 < \infty 
\end{cases} \tag{2.2}
\]

Where
\[ f_1(x_1, x_2) = f_{GE}(x_1; \alpha_{13}, \lambda) \int_{x_2} f_{GE}(x_2; \alpha_{2}, \lambda) \]
\[ = \alpha_{13} \alpha_{2} \lambda^{2} e^{-\lambda (x_1 + x_2)} [1 - e^{-\lambda x_2}]^{\alpha_{13} - 1} [1 - e^{-\lambda x_1}]^{\alpha_{2} - 1}, \]

and

\[ f_2(x_1, x_2) = f_{GE}(x_1; \alpha_{1}, \lambda) \int_{x_2} f_{GE}(x_2; \alpha_{33}, \lambda) \]
\[ = \alpha_{1} \alpha_{33} \lambda^{2} e^{-\lambda (x_1 + x_2)} [1 - e^{-\lambda x_2}]^{\alpha_{1} - 1} [1 - e^{-\lambda x_1}]^{\alpha_{33} - 1}. \]

Here \( c \) is the normalizing constant and \( c = \frac{\alpha_{123}}{\alpha_{12}} \). Therefore, the joint pdf of \((X_1, X_3)\) can be written as (2.2) and will be denoted by \( BBBGE(\alpha_1, \alpha_2, \alpha_3, \lambda) \).

In what follows the joint cdf corresponding to Equation (2.2), the marginal distributions of the BBBGE are presented.

**Proposition 2.1.** Let \((X_1, X_2) \sim BBBGE(\alpha_1, \alpha_2, \alpha_3, \lambda)\). The joint cdf is given as

\[ F_{X_1 X_2}(x_1, x_2) = \frac{\alpha_{123}}{\alpha_{12}} F_{GE}(x_1; \lambda, \alpha_1) F_{GE}(x_2; \lambda, \alpha_2) F_{GE}(x; \lambda, \alpha_3) \]
\[ - \frac{\alpha_3}{\alpha_{12}} F_{GE}(x; \lambda, \alpha_{123}); \]

Where \( x = \min(x_1, x) \). Moreover, the marginal cdfs are given by

\[ F_{X_1}(x_1) = \frac{\alpha_{123}}{\alpha_{12}} F_{GE}(x_1; \lambda, \alpha_1) - \frac{\alpha_3}{\alpha_{12}} F_{GE}(x_1; \lambda, \alpha_{123}) \]
\[ F_{X_2}(x_2) = \frac{\alpha_{123}}{\alpha_{12}} F_{GE}(x_2; \lambda, \alpha_2) - \frac{\alpha_3}{\alpha_{12}} F_{GE}(x_2; \lambda, \alpha_{123}) \]

**Proof:** The joint cdf given in (2.1) can be written as follows

\[ F_{MBBGE}(y_1, y_2) = \frac{\alpha_{12}}{\alpha_{123}} F_\alpha(y_1, y_2) + \frac{\alpha_1}{\alpha_{123}} F_\alpha(y_1) \]
\[ F_\alpha(y) = (1 - e^{-y})^{\alpha_{123}} \]

where \( F_\alpha(\ldots) \) and \( F_\alpha(\ldots) \) are the singular and the absolutely continuous part respectively. For \( y = \min(y_1, y_2) \),

\[ F_\alpha(y) = F_{GE}(y; \lambda, \alpha_{123}), \]

and

\[ F_\alpha(y_1, y_2) = \frac{\alpha_{123}}{\alpha_{12}} F_{GE}(y_1; \lambda, \alpha_1) F_{GE}(y_2; \lambda, \alpha_2) F_{GE}(y; \lambda, \alpha_3) \]
\[ - \frac{\alpha_3}{\alpha_{12}} F_{GE}(y; \lambda, \alpha_{123}). \]

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Once \( F_{X,Y}(y_1, y_2) = F_X(y_1, y_2) \), the result holds. The marginal cdfs are obtained simply.

**Proposition 2.2.** The marginal pdfs correspond to the cdf given in Proposition 2.1 are as follows

\[
f_{X_1}(x_1) = c F_{GE}(x_1; \lambda, \alpha_{123}) + c \frac{\alpha_3}{\alpha_{123}} f_{GE}(x_1; \lambda, \alpha_{123}), \quad x_1 > 0
\]

And

\[
f_{X_2}(x_2) = c F_{GE}(x_2; \lambda, \alpha_{23}) + c \frac{\alpha_3}{\alpha_{123}} f_{GE}(x_2; \lambda, \alpha_{123}), \quad x_2 > 0
\]

**Proof:** By apply \( f_{X_1}(x_1) = \frac{dF_{X_1}(x_1)}{dx_1} \) and \( f_{X_2}(x_2) = \frac{dF_{X_2}(x_2)}{dx_2} \), the results obtained.

Unlike those of the MOBGE distribution, the marginals of the BBBGE distribution are not GE distributions. If \( \alpha_3 \to 0^+ \), then \( X_1 \) and \( X_2 \) follow GE distributions and in this case, \( X_1 \) and \( X_2 \) become independent.

**Proposition 2.3.** Let \((X_1, X_2) \sim \text{BBBGE}(\alpha_1, \alpha_2, \alpha_3, \lambda)\). Then

i. the Stress-Strength parameter has the following form:

\[
R = P(X_1 < X_2) = \frac{\alpha_1}{\alpha_{12}},
\]

ii. \( \max(X_1, X_2) \sim \text{GE}(\alpha_{123}) \).

The BBBGE density may be unimodal depending on the values of \( \alpha_1, \alpha_2, \alpha_3 \) and \( \lambda \) that is \( f_{\text{BBBGE}}(x_1, x_2) \) is unimodal and the respective modes are

\[
\left\{ \frac{1}{\lambda} \ln(\alpha_{13}), \frac{1}{\lambda} \ln(\alpha_{2}) \right\} \text{ and } \left\{ \frac{1}{\lambda} \ln(\alpha_1), \frac{1}{\lambda} \ln(\alpha_{23}) \right\}.
\]

The median for the BBBGE distribution is obtained as

\[
-\frac{1}{\lambda} \left( 1 - \left( \frac{1}{2} \right)^{1/\alpha_{123}} \right).
\]
4 Maximum likelihood Estimation

In this Section, the maximum likelihood estimators (MLEs) of the unknown parameters of the BBBBB distribution are obtained. Suppose \( \{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\} \) is a random sample from \( BBBBB(\alpha_1, \alpha_2, \alpha_3, \lambda) \) distribution. Consider the following notation

\[
I_i = \{i; x_{1i} < x_{2i}\}, \quad I_i = \{i; x_{1i} > x_{2i}\}, \quad I = I_1 \cup I_2,
\]

\[
|I_1| = n_1, \quad |I_2| = n_2 \quad \text{and} \quad n_1 + n_2 = n.
\]

The log-likelihood function of the sample of size \( n \) is given by

\[
\ln L(\Theta) = \sum_{i \in I_1} \ln f_1(x_{1i}, x_{2i}) + \sum_{i \in I_2} \ln f_2(x_{1i}, x_{2i}) \quad (4.1)
\]

\[
\ln L(\Theta) = 2(n_1 + n_2) \ln \lambda + n_1 \ln(\alpha_1) + n_2 \ln(\alpha_2) + n_1 \ln(\alpha_3) \ln(\alpha_4 + \alpha_5)
\]

\[
+ n_2 \ln(\alpha_2 + \alpha_3) - \lambda \left[ \sum_{i=1}^{n_1} (x_{1i} + x_{2i}) + \sum_{i=1}^{n_2} (x_{1i} + x_{2i}) \right]
\]

\[
+ (\alpha_1 + \alpha_2 - 1) \sum_{i=1}^{n_1} \ln(1 - e^{-\lambda x_{1i}}) + (\alpha_2 - 1) \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda x_{2i}})
\]

\[
+ (\alpha_1 - 1) \sum_{i=1}^{n_1} \ln(1 - e^{-\lambda x_{1i}}) + (\alpha_2 - 1) \sum_{i=1}^{n_2} \ln(1 - e^{-\lambda x_{2i}})
\]

\[
(4.2)
\]

where \( \Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda) \).

On differentiating (4.2) with respect to \( \alpha_1, \alpha_2, \alpha_3 \) and \( \lambda \) and equating to zero, obtain the following likelihood equations are obtained.

\[
\frac{n_2}{\tilde{\alpha}_1} + \frac{n_1}{\tilde{\alpha}_1 + \tilde{\alpha}_3} + \sum_{i=1}^{n_1} \ln \left( 1 - e^{-\tilde{\lambda} x_{1i}} \right) + \sum_{i=1}^{n_2} \ln \left( 1 - e^{-\tilde{\lambda} x_{2i}} \right) = 0,
\]

\[
\frac{n_1}{\tilde{\alpha}_2} + \frac{n_2}{\tilde{\alpha}_2 + \tilde{\alpha}_3} + \sum_{i=1}^{n_1} \ln \left( 1 - e^{-\tilde{\lambda} x_{1i}} \right) + \sum_{i=1}^{n_2} \ln \left( 1 - e^{-\tilde{\lambda} x_{2i}} \right) = 0,
\]

\[
\frac{n_1}{\tilde{\alpha}_1 + \tilde{\alpha}_3} + \frac{n_2}{\tilde{\alpha}_2 + \tilde{\alpha}_3} + \sum_{i=1}^{n_1} \ln \left( 1 - e^{-\tilde{\lambda} x_{1i}} \right) + \sum_{i=1}^{n_2} \ln \left( 1 - e^{-\tilde{\lambda} x_{2i}} \right) = 0,
\]

and

\[
(\tilde{\alpha}_1 - 1) \sum_{i=1}^{n_1} \frac{x_{1i} e^{-\tilde{\lambda} x_{1i}}}{1 - e^{-\tilde{\lambda} x_{1i}}} + (\tilde{\alpha}_1 + \tilde{\alpha}_3 - 1) \sum_{i=1}^{n_2} \frac{x_{2i} e^{-\tilde{\lambda} x_{2i}}}{1 - e^{-\tilde{\lambda} x_{2i}}} + \frac{2(n_1 + n_2)}{\tilde{\lambda}} - \sum_{i=1}^{n_1} (x_{1i} + x_{2i})
\]

\[
(\tilde{\alpha}_2 - 1) \sum_{i=1}^{n_1} \frac{x_{2i} e^{-\tilde{\lambda} x_{2i}}}{1 - e^{-\tilde{\lambda} x_{2i}}} + \frac{2(n_1 + n_2)}{\tilde{\lambda}} - \sum_{i=1}^{n_2} (x_{1i} + x_{2i})
\]
\[ a_{44} = -\frac{\partial^2 \ln L}{\partial \lambda^2} = (\hat{\alpha}_1 + \hat{\alpha}_3 - 1) \sum_{r=1}^{n_1} \frac{x_r^2 e^{-\lambda x_r}}{(1 - e^{-\lambda x_r})^2} + (\hat{\alpha}_1 - 1) \sum_{r=1}^{n_2} \frac{x_r^2 e^{-\lambda x_r}}{(1 - e^{-\lambda x_r})^2} \]
\[ + (\hat{\alpha}_2 + \hat{\alpha}_3 - 1) \sum_{r=1}^{n_3} \frac{x_r^2 e^{-\lambda x_r}}{(1 - e^{-\lambda x_r})^2} + (\hat{\alpha}_2 - 1) \sum_{r=1}^{n_4} \frac{x_r^2 e^{-\lambda x_r}}{(1 - e^{-\lambda x_r})^2} + \frac{2(n_1 + n_2)}{\rho^2}. \]

Now, to obtain the asymptotic confidence intervals of \( \lambda, \alpha_1, \alpha_2 \) and \( \alpha_3 \). The asymptotic normality results can be stated as follows
\[ \sqrt{n} [(\hat{\lambda} - \lambda), (\hat{\alpha}_1 - \alpha_1), (\hat{\alpha}_2 - \alpha_2), (\hat{\alpha}_3 - \alpha_3)] \rightarrow N(0, I^{-1}(\Theta)) \text{ as } n \rightarrow \infty \] (4.4)
where \( I^{-1}(\Theta) \) is the variance-covariance matrix, \( \hat{\Theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\lambda}) \) and \( \Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda) \). Since \( \Theta \) is unknown in (4.4), \( I^{-1}(\hat{\Theta}) \) is estimated by \( I^{-1}(\hat{\Theta}) \).

5 Simulation Results

In this Section, a simulation experiment is presented in which the estimation of the parameters of the BBBGGE distribution are evaluated. The simulations were performed using the Mathcad program, the number of the replications \( R = 1000 \).

The evaluation of the point estimation was performed based on the following quantities for each sample size: the Average Estimates (AE), the Mean Squared Error, (MSE) are estimated from \( R \) replications and the coverage rate of the 95% confidence interval for \( \lambda, \alpha_1, \alpha_2 \) and \( \alpha_3 \), the sample size is chosen at \( n = 20, 40, 60 \) and 100, and considered some values for the parameters \( \lambda, \alpha_1, \alpha_2 \) and \( \alpha_3 \).

It can be seen from Table 1 that the estimates are slightly positively biased and that the MSE decreases as the sample size increases, as expected. The estimates are close to the true values. Also the coverage probabilities are close to the nominal level. These results indicate that the proposed model and the asymptotic approximation work well under the situation where no censoring occurs.
Table 1: The average estimates (AE) of α₁, α₂, α₃, and λ for BBBGE distribution

<table>
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<th>95% CI Coverage</th>
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6. Conclusion

In this paper the absolutely continuous bivariate model following the approach of Block and Basu (1974) has been introduced. That obtained from the Marshal – Olkin bivariate generalized exponential model by removing the singular part. The BBBGE model has an absolutely continuous probability density function. The moment generating function for the BBBGE distribution has been obtained. The exact forms for the distribution of sum; ratio and product of dependent variables follows the BBBGE distribution have been derived. The maximum likelihood estimates for the four unknown parameters and their approximate variance- covariance matrix have been obtained. Finally some a numerical illustration has been performed to see the performances of the MLEs.
References


