Forms of the Moments of AR (P) Model with Missing Observations

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Abstract

In this paper, the Moments of AR (p) Model with missing observations, will be derived and a special cases will also be introduced.

Keywords and phrases: the Mean, the Variance, AR (P) Model with missing observations.

1. Introduction

Parzen (1963) introduced the time series model with missing observations as a specific case of an amplitude modulated stationary process. He observed that the data $\{y_1, y_2, \cdots\cdots, y_p\}$ can be represented as,

$$x_k = \eta + \sum_{i=1}^{p} \rho_i x_{k-i} + \varepsilon_k \qquad \varepsilon_k \sim iid\left(0, \sigma^2\right) \tag{1}$$

$$y_k = a_k x_k \qquad k = 1, 2, \cdots\cdots, p \tag{2}$$

$$\mathrm{E}(\varepsilon_k) = 0 \tag{3}$$

$$\mathrm{E}(y_k) = a_k \mathrm{E}(x_k) \tag{4}$$

$$\mathrm{E}(x_{k-i} \varepsilon_k) = 0 \tag{5}$$

Where, $\{a_k, k = 1,2,\cdots\cdots,p\}$ represent the state of observation such that,

$$a_k = \begin{cases} 1 & \text{if } x_k \text{ is obsrved,} \\ 0 & \text{O/W} \end{cases}$$

Typical examples of $\{a_k\}$ are a stochastic markov process and periodically deterministic case. He assumed that $\{\varepsilon_k\}$ and $\{a_k\}$ are independent if $\{a_k \neq 1\}$ and $\{\varepsilon_k\}$ is a stochastic process.

If $\{x_k\}$ is a stationary process, which is the start from the infinite past.

Dunsmuir and Robinson (1981) suggested the estimator of AR (1) model with missing observations. Mahmoud, et al (2012) suggested the general form for the moments of AR (P) Model where the data is complete. Mahmoud, et al (2016) derived the estimate parameter of AR (1) and AR (2) model with missing observations.

2. Properties of the Autoregressive Models

The properties of the autoregressive models with intercept term of AR (p) will be introduced in the following section

2.1 The Mean of AR (p)

The AR (p) with the constant term by using equation (1) and (2) takes the form:

$$y_k = \eta a_k + \sum_{i=1}^{p} \rho_i a_k x_{k-i} + a_k \varepsilon_k \tag{6}$$

By taking the expectation of equation (6) and using equation (3),(4) and (5) we will get

$$\mathrm{E}(y_k) = \frac{\eta a_k}{1 - \sum_{i=1}^{p} \rho_i} \qquad \sum_{i=1}^{p} \rho_i \neq 1 \tag{7}$$

From equation (7) we can get some Special cases as follows:

a) **The mean of AR (1), will be**

$$E(y_k)=\frac{\eta a_k}{1-\rho_1}, \qquad \rho_1 \neq 1$$

Where p=1. In the case of complete data , $a_k=1$, we obtained the mean of AR(1) as refer by Robert and David (2017).

b) **The mean of AR (2)**

$$E(y_k)=\frac{\eta a_k}{1-\rho_1-\rho_2}, \qquad \rho_1+\rho_2 \neq 1$$

Where p=2. In the case of complete data , $a_k=1$, we obtained the mean of AR(2) as refer by Robert and David (2017).

c) **The mean of AR (3)**

$$E(y_k)=\frac{\eta a_k}{1-\rho_1-\rho_2-\rho_3}, \qquad \rho_1+\rho_2+\rho_3 \neq 1$$

Where p=3. In the case of complete data , $a_k=1$, we obtained the mean of AR(3) as refer by Mahmoud (2012).

2.2 The Variance of AR (p)

By squaring equation (6) and taking the expectation, we will get

$$y_k^2=\left[\eta a_k+\sum_{i=1}^{p}\rho_i a_k x_{k-i}+a_k\varepsilon_k\right]^2$$

$$E(y_k^2)=\eta^2 a_k^2+E\left(\sum_{i=1}^{p}\rho_i a_k x_{k-i}\right)^2+a_k^2 E(\varepsilon_k^2)+2\eta a_k E\left(\sum_{i=1}^{p}\rho_i a_k x_{k-i}\right)$$
$$+2\eta a_k^2 E(\varepsilon_k)+2a_k E(\varepsilon_k)\sum_{i=1}^{p}\rho_i a_k x_{k-i}$$

$$E(y_k^2)=\eta^2 a_k^2+\left(\sum_{i=1}^{p}\rho_i E(a_k x_{k-i})\right)^2+a_k^2\sigma^2+2\eta a_k\left(\sum_{i=1}^{p}\rho_i E(a_k x_{k-i})\right)$$

$$E(y_k^2)=\eta^2 a_k^2+E(y_k^2)\sum_{i=1}^{p}\rho_i^2++a_k^2\sigma^2+2\eta a_k E(y_k)\left(\sum_{i=1}^{p}\rho_i\right)$$

$$E\left(y_k^2\right)=\frac{\eta^2 a_k^2+a_k^2\sigma^2+2\frac{\eta^2 a_k\left(\sum_{i=1}^{p}\rho_i\right)}{1-\sum_{i=1}^{p}\rho_i}}{1-\sum_{i=1}^{p}\rho_i^2} \qquad (8)$$

$$\operatorname{var}(y_k)=E\left(y_k^2\right)-\left[E(y_k)\right]^2=\frac{\eta^2 a_k^2+a_k^2\sigma^2+2\frac{\eta^2 a_k\left(\sum_{i=1}^{p}\rho_i\right)}{1-\sum_{i=1}^{p}\rho_i}}{1-\sum_{i=1}^{p}\rho_i^2}-\frac{\eta^2 a_k^2}{\left(1-\sum_{i=1}^{p}\rho_i\right)^2}$$

$$\operatorname{var}(y_k)=\frac{\eta^2 a_k^2\left(1-\sum_{i=1}^{p}\rho_i\right)^2+a_k^2\sigma^2\left(1-\sum_{i=1}^{p}\rho_i\right)^2+2\eta^2 a_k\left(\sum_{i=1}^{p}\rho_i\right)\left(1-\sum_{i=1}^{p}\rho_i\right)-\eta^2 a_k^2\left(1-\sum_{i=1}^{p}\rho_i^2\right)}{\left(1-\sum_{i=1}^{p}\rho_i^2\right)\left(1-\sum_{i=1}^{p}\rho_i\right)^2}$$

$$\operatorname{var}(y_k)=\frac{a_k^2\sigma^2\left(1-\sum_{i=1}^{p}\rho_i\right)^2-\eta^2 a_k^2\left(\sum_{i=1}^{p}\rho_i\right)^2+\eta^2 a_k^2\sum_{i=1}^{p}\rho_i^2}{\left(1-\sum_{i=1}^{p}\rho_i^2\right)\left(1-\sum_{i=1}^{p}\rho_i\right)^2}=\frac{a_k\sigma^2\left(1-\sum_{i=1}^{p}\rho_i\right)^2-\eta^2 a_k^2\sum_{i=1}^{p}\rho_i^2-2\eta^2 a_k^2\sum_{i=1}^{p}\sum_{j=1}^{p}\rho_i\rho_j+\eta^2 a_k^2\sum_{i=1}^{p}\rho_i^2}{\left(1-\sum_{i=1}^{p}\rho_i^2\right)\left(1-\sum_{i=1}^{p}\rho_i\right)^2}$$

$$Var(y_k)=\frac{a_k^2\sigma_\varepsilon^2}{1-\sum_{i=1}^{p}\rho_i^2}-2\eta^2 a_k^2\frac{\sum_{i=1}^{p}\sum_{j=1}^{p}\rho_i\rho_j}{\left(1-\sum_{i=1}^{p}\rho_i^2\right)\left(1-\sum_{i=1}^{p}\rho_i\right)^2} \qquad i\neq j \qquad (9)$$

By assuming that

$$\sum_{i=1}^{p}\rho_i^2\neq 1 \quad \text{and} \quad \sum_{i=1}^{p}\rho_i\neq 1 \quad \text{and} \quad \rho_{-1}=0$$

From equation (9) we can get some Special cases as follows:

a) **The Variance of AR (1)**

$$Var(y_k)=\frac{a_k^2\sigma_\varepsilon^2}{\left(1-\rho_1^2\right)} \qquad -1<\rho_1<1$$

Where p=1. In the case of complete data , $a_k=1$, we obtained the mean of AR(1) as refer by Robert and David (2017).

b) **The Variance of AR (2)**

$$Var(y_k)=\frac{a_k^2\sigma_\varepsilon^2}{1-\rho_1^2-\rho_2^2}-\frac{2\eta^2a_k^2\rho_1\rho_2}{\left(1-\rho_1^2-\rho_2^2\right)\left(1-\rho_1-\rho_2\right)^2}$$

$$\sum_{i=1}^{2}\rho_i^2\neq 1 \text{ and } \sum_{i=1}^{2}\rho_i\neq 1 \text{ and } \rho_{-1}=0$$

Where p=2. In the case of complete data , $a_k=1$, we obtained the mean of AR(2) as refer by Robert and David (2017).

c) **The Variance of AR (3)**

$$Var(y_k)=\frac{a_k^2\sigma_\varepsilon^2}{1-\rho_1^2-\rho_2^2-\rho_3^2}-\frac{2\eta^2a_k^2\rho_1\rho_2+2\eta^2a_k^2\rho_1\rho_3+2\eta^2a_k^2\rho_2\rho_3}{\left(1-\rho_1^2-\rho_2^2-\rho_3^2\right)\left(1-\rho_1-\rho_2-\rho_3\right)^2}$$

$$\sum_{i=1}^{3}\rho_i^2\neq 1 \text{ and } \sum_{i=1}^{3}\rho_i\neq 1 \text{ and } \rho_{-1}=0$$

Where p=3. In the case of complete data , $a_k=1$, we obtained the mean of AR(3) as refer by Mahmoud (2012)

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