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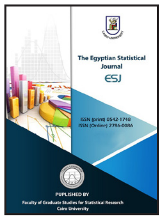

Bayesian Estimation of the Exponentiated Fréchet Distribution Parameters

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A B S T R A C T

This article discusses the Bayesian estimators of the parameters of Exponentiated Fréchet distribution under squared error loss function, elative error loss function and LINEX loss function using the non-informative and uniform priors. The performance of the proposed estimators has been compared based on their simulated risk and the mean square error (MSE)

Keywords:

Frechet distribution; Exponentiated distribution; Bayesian Estimation.

1. **Introduction**

Gupta et (1998) proposed family of exponentiated distributions generated by Lahmann alternatives with cumulative distributions of the from $F(\chi) = [G(\chi)]^a$ or $F(\chi) = 1 - [1-G(\chi)]^a$ to model failure time data, where G(x) is the baseline distribution function and a is a positive real number. This model gives rise to monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic. Nadarajah and Kotz (2003) introduced the exponentiated Fréchet distribution, EFD, as a member of the exponentiated family with cumulative distribution function

$$F(x;a,b,p) = [1\text{-}\exp(-(\tfrac{p}{x})^b]^a, \qquad x > 0,\ b>0 \text{ and } p>0. \tag{1}$$

The corresponding pdf is

$$f(x;a,b,p) = abp^b[1 - exp\{-(\tfrac{p}{x})^b\}]^{a-1}x^{-(b+1)}\, exp\{-(\tfrac{p}{x}\}^b\} \tag{2}$$

As a special case at =1 the EFD reduces to the standard Fréchet distribution which has many applications in earthquakes, floods, rainfall and wind speeds, see Kotz and Nadarajah (2000) for more applications.

The inferential procedures for lifetime models are often developed using squared error loss function. The use of the squared error loss is well justified when the loss is symmetric in nature, its use is very popular because of its mathematical simplicity. But in life testing and reliability, the nature of losses is not always sympatric and hence the use of symmetric losses is unacceptable in many situations. It is because of this fact that Varian (1975) introduced LINEX loss function as asymmetric loss function. In this note, we concerning with the Bayes point estimation of the three parameters EFD whit respect to the squared loss, relative loss and LINEX loss functions using non-informative and gamma priors for the parameters.

The rest of this paper organized as follows; section 2 presents the maximum likelihood equations for the EFD. An integral of the Bayes estimations based on different loss functions are derived. The maximum likelihood equations and the Bayes estimation integral forms are not tractable analytically, so suitable numerical methods are used in section 3 to show the performance of the proposed estimators using a simulation study based on the simulated risk and the MSE. The last section contains a brief conclusion on the use of the estimation.

2. Estimation

Consider the random sample X= ($x_1, x_2, \ldots, x_n$) of the EFD in (1). The likelihood function of the EFD is:

$$L(X|a,b,p) = a^n b^n p^{bn} \overset{n}{\underset{i=1}{\pi}} \left(x_i^{-(b+1)} exp\{-(\frac{p}{x_i})^b\}[1-exp\{-(\frac{p}{x_i})^b\}]^{a-1} \right) \qquad (3)$$

2.1 Maximum Likelihood Estimation

By equation the first derivatives of the log-likelihood function with respect to a, b, p to zero, the maximum likelihood equations take the form,

$$\frac{nb}{p} + (a-1)bp^{b-1}\sum_{i=1}^{n} \frac{exp\{-(\frac{p}{x_i})^b\}}{x_i{}^b[1-exp\{-(\frac{p}{x_i})^b\}]} - bp^{b-1}\sum_{i=1}^{n} x_i{}^{-b} = 0$$

$$\frac{n}{b} + n\log p + (a-1)p^{b-1}\sum_{i=1}^{n} \frac{log(\frac{p}{x_i})exp\{-(\frac{p}{x_i})^b\}}{x_i{}^b[1-exp\{-(\frac{p}{x_i})^b\}]} - \sum_{i=1}^{n} log\, x_i - \sum_{i=1}^{n} log(\frac{p}{x_i})(\frac{p}{x_i})^b = 0$$

$$\frac{n}{a} + \sum_{i=1}^{n} log[1-exp\{-(\frac{p}{x_i})^b\}] = 0$$

(4)

Solving equations (4) simultaneously yields the maximum likelihood estimates for a, b, p. It can be noted that equations (4) are not easy to solve implicitly, however numerical methods are available.

2.2 Bayes Estimations

Consider a joint prior function Π(a,b,p) for the parameters a, b, p of EFD ,the joint posterior distribution is :

$$\pi(a,b,p|X) = \frac{L(X|a,b,p)\pi(a,b,p)}{\int\int\int L(X|a,b,p)\pi(a,b,p)} \qquad (5)$$

The marginal posterior is obtained by integrating the joint posterior distribution with respect to the other two parameters and hence, taking ϴ=(a,b,p),the marginal posterior for θ_i is

$$\pi(\theta_i|X) = \int\int \pi(\theta|X)\, d\theta_j\, d\theta_k \qquad , \mathbf{i}=1, 2, 3 \qquad (6)$$

Where j,k=1,2,3 and j≠i≠k.

2.2.1 Bayes Estimations Under Symmetric Loss Function

The squared error loss function and the relative error loss function are special cases of the weight function

$$l(\theta,\hat{\theta}) = \sum_{i=1}^{3} w_i(\theta_i)(\theta_i - \hat{\theta}_i)^2$$

when $w_i(\theta_i)$=1 for i=1,2,3 and $w_i(\theta_i)=1/\theta^2{}_i$ for i=1,2,3,respectively.It is easy to show that the Bayes estimator $\hat{\theta}_i$ of θ_i can be obtained as

$$\hat{\theta}_i = E[\theta_i \; w_i(\theta_i) | X] / E[w_i(\theta_i) | X] \qquad (7)$$

where the expectations are with respect to the marginal posterior of θ_i at (6).

Assuming independent non-informative priors of θ_1, θ_2 , θ_3, the joint non-informative prior is

$$\pi_1(\theta_1,\theta_2,\theta_3) = \frac{1}{\theta_1\theta_2\theta_3} \qquad ,\theta_1,\theta_2,\theta_3>0 \qquad (8)$$

and assuming the uniform priors is

$$\pi_2(\theta_1,\theta_2,\theta_3) = \frac{1}{a_i - b_i} \qquad \text{for} \quad a_i \prec \theta_i \prec b_i, i=1,2,3 \qquad (9)$$

Using equations (5) to (9), the Bayes estimators based on the squared error loss function are

$$\hat{\theta}_1(ij) = E[\theta_i | X] = \frac{\int\int\int \theta_i L(X|\theta)\pi_j(\theta)d\theta}{\int\int\int L(X|\theta)\pi_j(\theta)d\theta}$$

where i=1,2,3 and j=1,2. The Bayes estimators based on the relative error function is $\hat{\theta}_2(ij) = I_{(ij)} / J_{(ij)}, i = 1,2,3$ and j=1,2, where

$$I_{(ij)} = \frac{\int\int\int \theta^{-1}{}_i L(X|\theta)\pi_j(\theta)d\theta}{\int\int\int L(X|\theta)\pi_j(\theta)d\theta}$$

and

$$J_{(ij)} = \frac{\int\int\int \theta^{-2}{}_i L(X|\theta)\pi_j(\theta)d\theta}{\int\int\int L(X|\theta)\pi_j(\theta)d\theta}$$

where $L(X|\theta)$ is as in (3).It may be noted that these Bayes estimators cannot be reduced to closed forms, however, we propose to use Tierney and Kadane (1986) method as an accurate approximation for the integrals $I_{(ij)}$ and $J_{(ij)}$.These integral are approximated using the following formula

$$I_{(ij)} \cong \left(\left|\sum\nolimits^*_{1(ij)}\right| / \left|\sum\nolimits_{1j}\right|\right)^{1/2} \exp[n\{L_1^*{}_{(ij)}(\theta^*) - L_{1j}(\hat{\theta})\}] \qquad (10)$$

where

$$L_{1j}(\theta) = n^{-1}[\log L(X|\theta) - \log \pi_j(\theta)]$$

and

$$L^*{}_{1(ij)}(\theta) = L_{1j}(\theta) - n^{-1}\log\theta_i$$

The value of $\hat{\theta}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)$ maximizes $L_{1j}(\theta)$,i.e. Solves the equations $\frac{\partial L_{1j}(\theta)}{\partial \theta_i}=0$ .also $\theta^*=(\theta_1^*,\theta_2^*\theta_3^*)$ maximizes $L^*_{1(ij)}(\theta)$.The matrices $\sum_{1j} and \sum^*_{1(ij)}$ are the negatives of the inverse of the second order Hessian matrices of $L_{1j}(\theta)$ and $L^*_{1(ij)}(\theta)$ at $\hat{\theta}$ and θ^*, respectively. The integral $J_{(ij)}$ is approximated as $I_{(ij)}$ in (10) with $L^*_{2(ij)}(\theta)=L_{1j}(\theta)-2n^{-1}\log\theta_i$ instead of $L^*_{1(ij)}(\theta)$ and with its corresponding $\sum^*_{2(ij)}$.

2.2.2. Bayes estimators Under LINEX loss function

The LINEX loss function was introduced by Varian (1975) and is given by

$$L(\theta,\hat{\theta})=b_0\ [\exp\{\ A_0\ ((\hat{\theta}/\theta)\ -1)\}-\ A_0\ ((\hat{\theta}/\theta)\ -1)-1] \tag{11}$$

Where A_0 and b_0 are constants and $b_0>0$, $A_0>0$. The LINEX loss function was employed for estimation by several authors [Zellner (1986), Calabria and Pulcini (1996) and El-Helbawy (2004)]. The Bayes estimator of θ under the Modified linear exponential (MLINEX) loss function and that of a,b,p under the linear Exponential (LINEX) loss function when considering various priors will be derived.

2.2.2.1 Bayes estimator ($\hat{\theta}_{iL}$)of θ under (MLINEX) loss function

If a,b,p are both unknown and any prior $\pi_i(a,b,p)$ is considered, the Bayes estimator $\hat{\theta}_{iL}$ under the Modified LINEX loss function.

$$E[L(\hat{\theta},\theta|x)]=\int_0^\infty L(\hat{\theta},\theta)\ \pi_i(\theta|x)da$$

Considering the Modified LINEX function, it follows that

$$E[L(\hat{\theta},\theta|x)]=b0\int_0^\infty [\exp\{A0(\hat{\theta}/\theta-1)\}-A0(\hat{\theta}/\theta-1)-1]\ \pi_i(\theta|x)\,d\theta, \tag{12}$$

Where $\pi_i(a\,|\,x)$ is the marginal distribution of θ. $\hat{\theta}$ which minimizes $E[L(\hat{\theta},\theta)|x]$ satisfies the following equation.

$$E[\theta^{-1}e^{A_0(\hat{\theta}/\theta)}|x]=e^{A_0}\ E[1/\theta|x]$$

Where both expectations are with respect to the marginal distribution of θ. Evaluting of $E[\theta^{-1}e^{A(\hat{\theta}/\theta)}]$ and $E[1/\theta|x]$ can be done numerically.

3. A Simulation Study

In this section a simulation study is presented to illustrate the application of the various theoretical results developed in the previous sections.

In subsection (3.1) numerical results are given for the case where three parameters is unknown see [Mark .E..Jonhson (1984)],and is presented graphics for the MSE under the others loss functions when the sample size is given (10,20,30,40,50).The last subsection (3.2)is presented the comments for all results obtained.

3.1 Simulation study when a, b, p are unknown:

In this example, the Bayes estimators for a under relative error loss function are compared with those using the squared-error loss function and the (LINEX) loss function under two different priors (non-informative and uniform priors) Al-Hussaini, E.K. and Jaheen, Z.F. (1995).

Samples are generated from the exponentiated Fréchet distribution for three different values of a (true values) namely a, b, p=1,2,3. Corresponding to each value, for b=2,3,4, p=2,3,4 are considered. A simulation study of size (ns 1000) is considered. The sample size is chosen n=10,20,30,40,50.

the squared deviation $(\hat{\theta} - \theta)^2$ are computed for the different estimates. The results are repeated (1000) times. An estimate of the Mean squared Error MSE is computed by the average of the squared deviations over the ns repetitions. The prior parameters chosen for the uniform prior are a_i=1 and b_i =5. The results are display in the appendix.

3.2 Comments on graphics:

The method of Bayesian estimation for estimating the shape and scale parameters of the EFD is studied in Mathcad 2010. The bayes estimators for a under relative error loss function are compared with those using the squard-error loss function and the LINEX loss function under two different priors (non-informative and uniform priors).

-A simulation study of size 1000 is considered. The sample size n=10,20,30,40,50.

In figure (1.1), (1.3), (1.5): is shown the MSE of a=2 under noninformative prior. For three different values of a (true values) namely a =1,2,3 corresponding to each value, for b=2,3,4 and p=2,3,4. We note that: The bayes estimators of a,b,p under the LINEX error loss function perform better than the corresponding results under the squared error loss function better than the corresponding results under the relative error loss function.

In figure (1.2), (1.4), (1.6):

is shown the MSE of a=2 under uniform prior. For three different values of a (true values) namely a=1,2,3 corresponding to each value, for b=2,3,4 and p=2,3,4. We note that: The bayes estimators of a,b,p under the LINEX error loss function perform better than the corresponding results under the squared error loss function better than the corresponding results under the relative error loss function. The bayes estimators of a,b,p using the uniform prior perform better than those using the non-informative prior. This indicated that more prior information leads to more accurate results.

- It is observed that estimates for a,b,p perform better where the sample size n is increased. Comparing estimates of tables when n=50.It is observed that estimates for a=1, b=2, p=2 perform better than those using a=2, b=3, p=3 perform better than those using a=3, b=4, p=4.

3.3 Conclusion:

This paper has been obtained Bayes and maximum likelihood estimators for the three parameters exponentiated Fréchet distribution. We have obtained Bayesian estimators of the parameters of the Exponentiated Fréchet distribution under squared error loss function, relative error loss function and LINEX loss function using the non-informative and uniform priors. The performance of the proposed estimators has been compared based on their simulated risk and the mean square error (MSE).

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REFERENCES

1] Al-Hussaini, E.K. and Jaheen, Z.F. (1995). Bayesian prediction Bounds for the Burr Type XII failure Model. Commun. Statist.-Theory Meth. 24(7), 1829-1842.

2] Calabria, R. and Rulicini, G. (1996). Point Estimation under Asymmetric loss Functions for left Truncated exponential Samples. Commun. Statist.-Theory Meth., 25(3), 585-600.

3] El-Helbawy, A.A. (2004). The Inverse Weibull Distribution As A life Time Model. Unpublished pH.D. Dissertation, Faculty of Commerce, Al-Azhar University (Girls Branch).

4] Kotz,S. and Nadarajah,S.(2000).Extreme value Distributions:Theory and applications.London:Imperial College press.

5] Mark E. Johnson (1984): Multivariate statistical simulation. John Wiley & Sons.

6] Nadarajah,S.and Kotz,S.(2003).On the exponentiated exponential distribution, to appear in statistica.

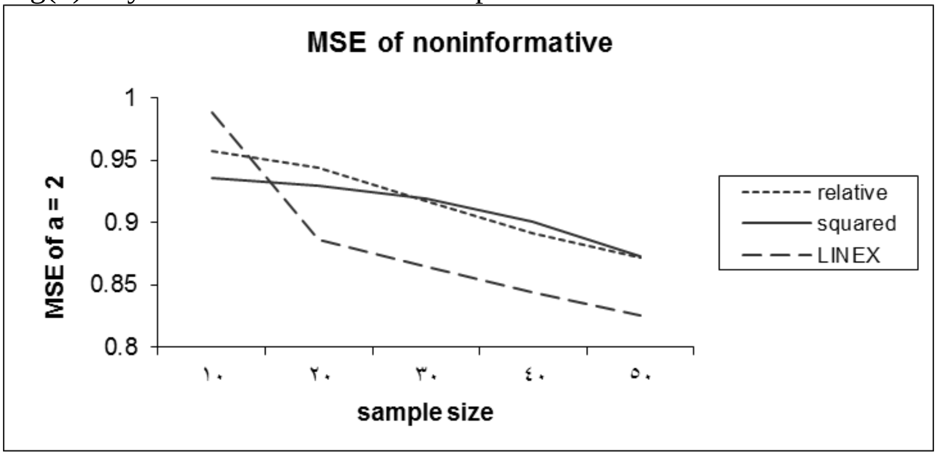
7] Tierney, L. and Kadane, J. B. (1986): Accurate approximations for posterior moments and marginal densities. JASA, 81, 393, 82-86.

8] Varian, Hal R. (1975): A Bayesian Approach to real estate assessment. In Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage, eds. Stephen E. Fienberg and Arnold Zellner, Amsterdam: North-Holland, 195-208.

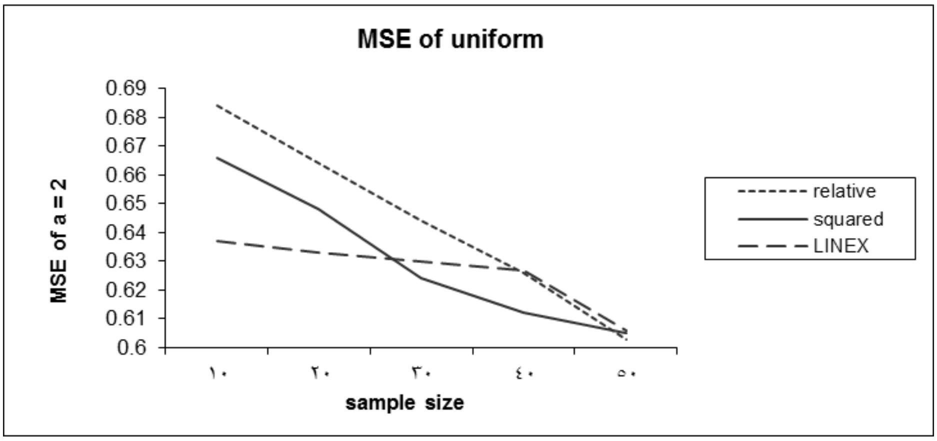
9] Zellner, A. (1986). Bayesian Estimation and prediction using asymmetric loss functions, JASA, 81, 446-451.

Appendix

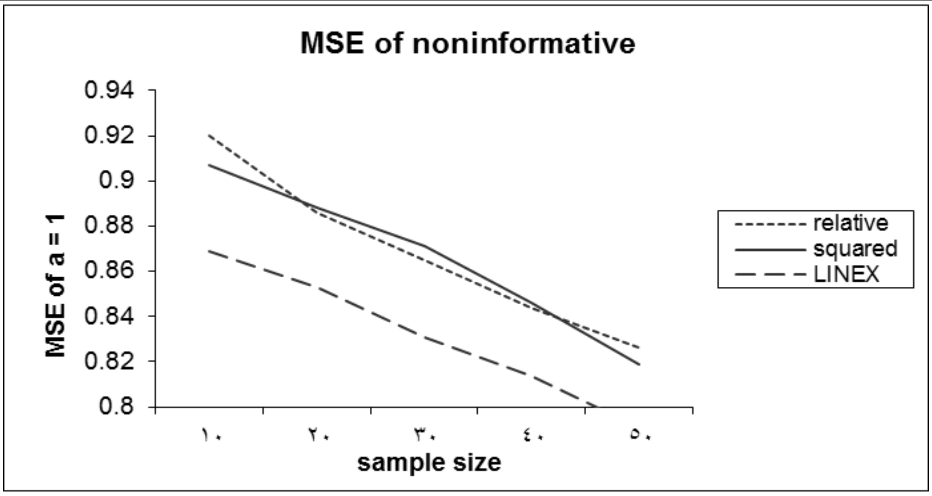
Fig(1) Bayesian estimation of the Exponentiated Frechet distribution when a,b,p are unknown$a$



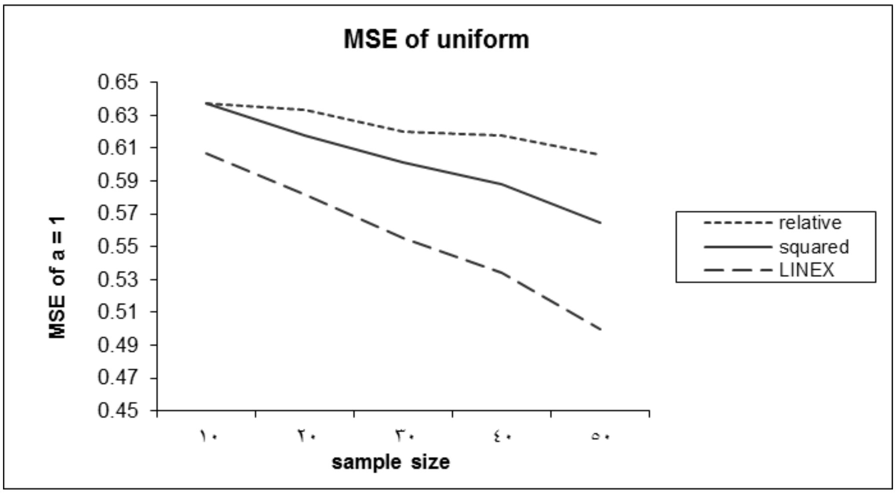
Fig(1.1)The MSE of a=2 under noninformative prior



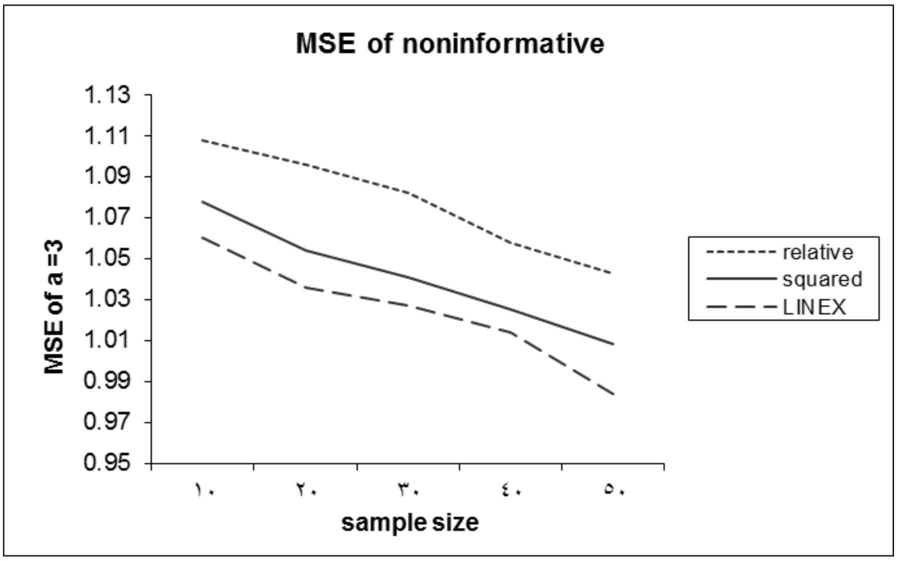
Fig(1.2)The MSE of a=2 under uniform prior



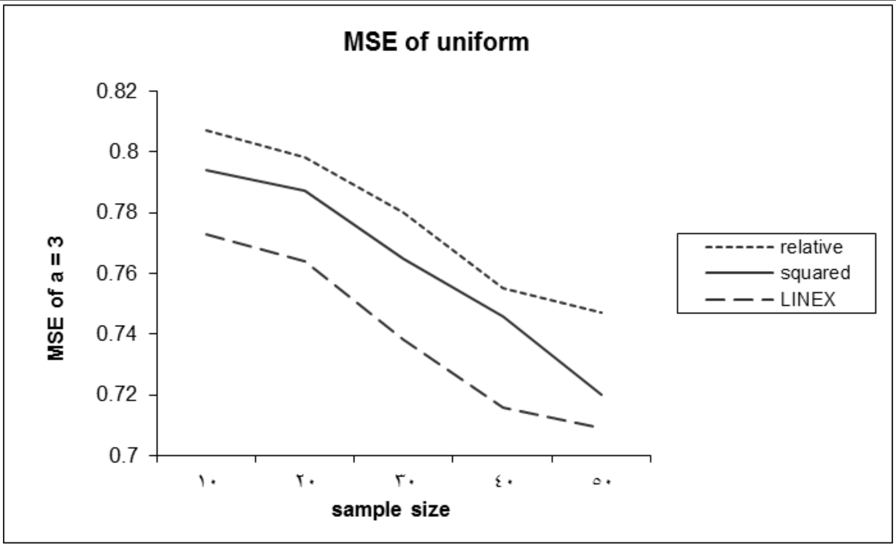
Fig(1.3)The MSE of a=1 under noninformative prior



Fig(1.4)The MSE of a=1 under uniform prior

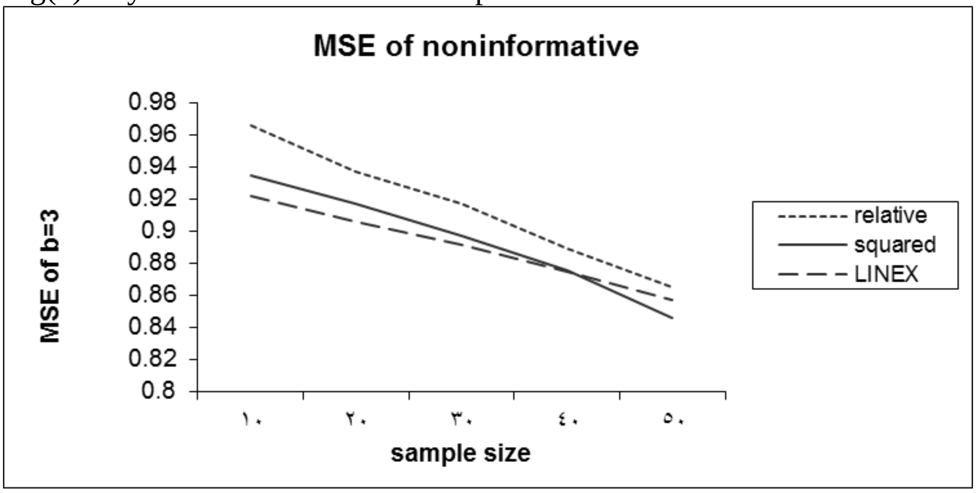


Fig(1.5)The MSE of a=3 under noninformative prior

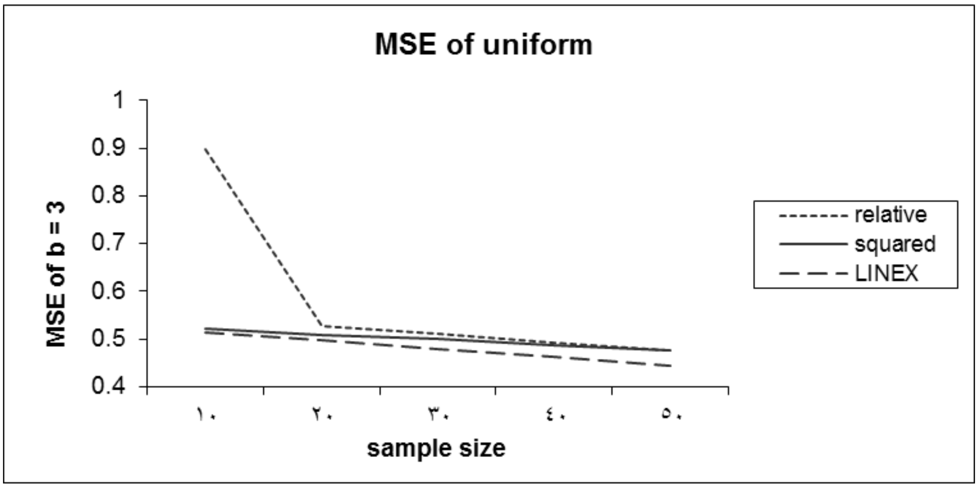


Fig(1.6)The MSE of a=3 under uniform prior

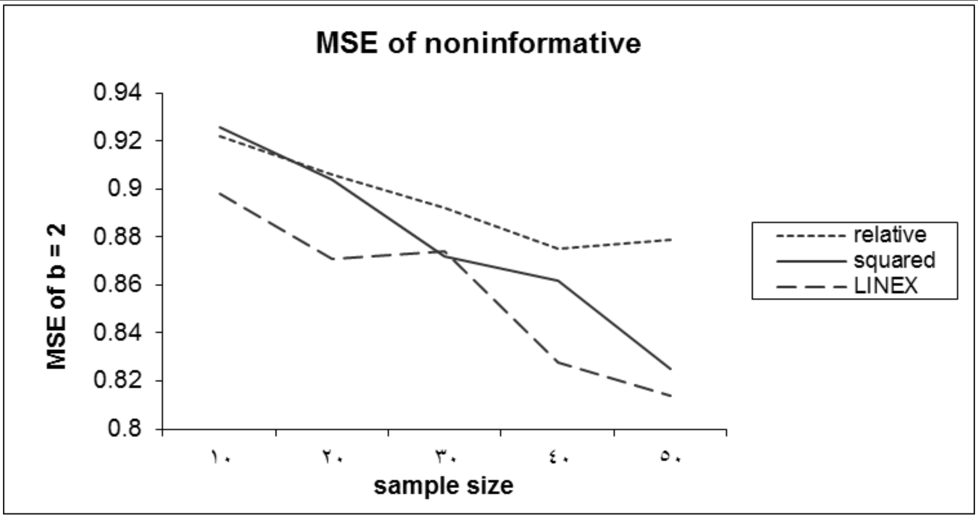
Fig(2) Bayesian estimation of the Exponentiated Frechet distribution when a,b,p are unknown



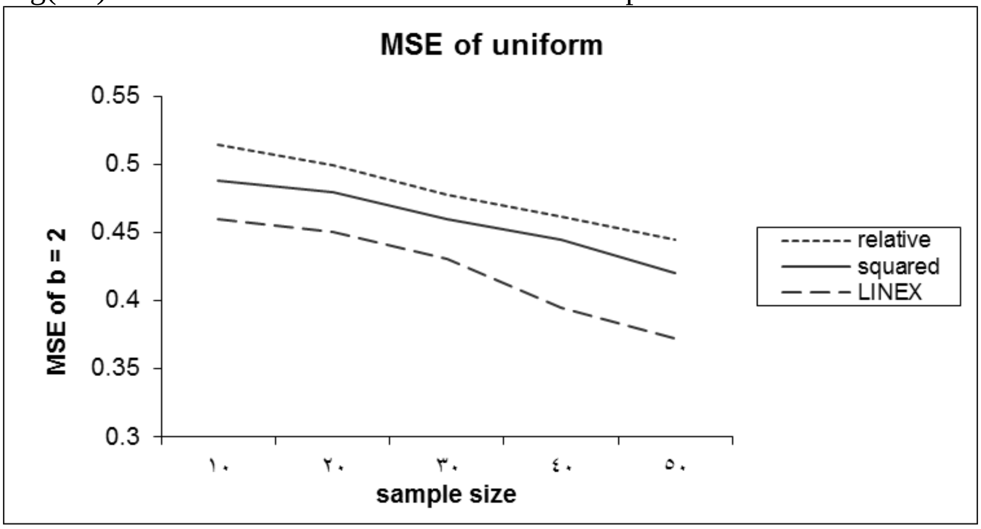
Fig(2.1)The MSE of b=3 under noninformative prior



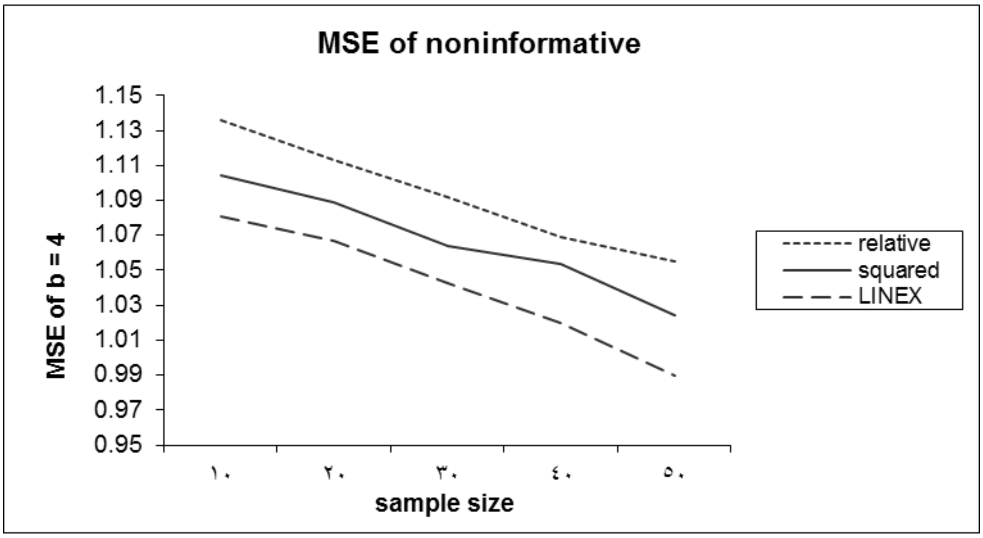
Fig(2.2)The MSE of b=3 under uniform prior



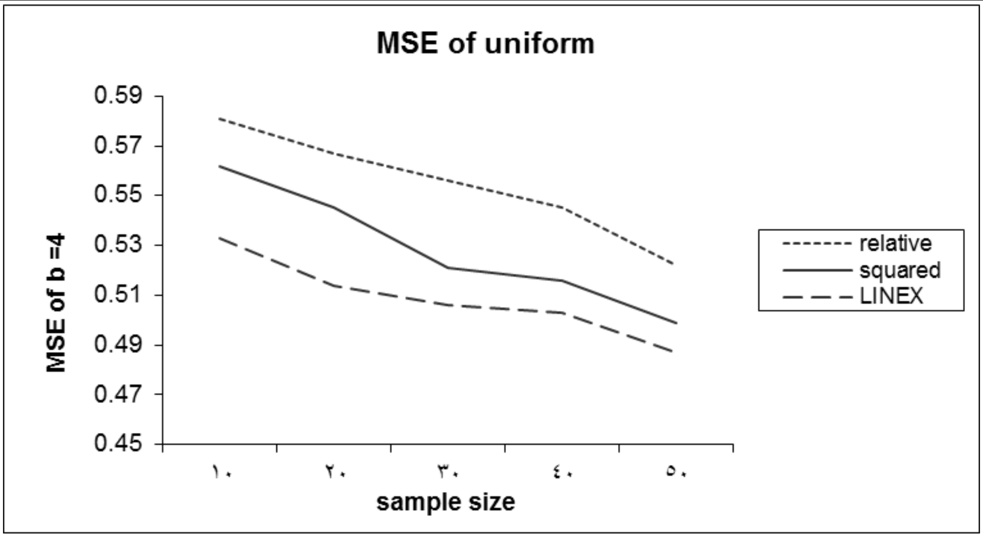
Fig(2.3)The MSE of b=2 under noninformative prior



Fig(2.4)The MSE of b=2 under uniform prior

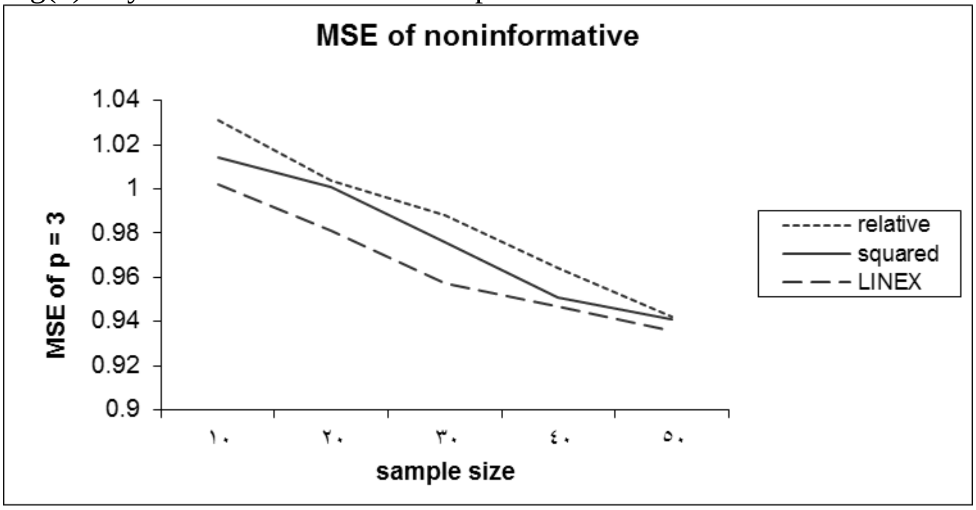


Fig(2.5)The MSE of b=4 under noninformative prior

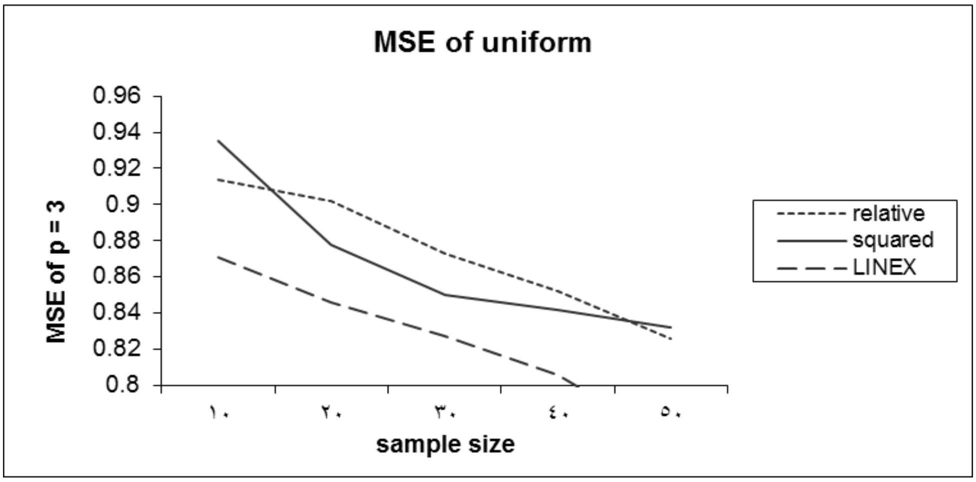


Fig(2.6)The MSE of b=4 under uniform prior

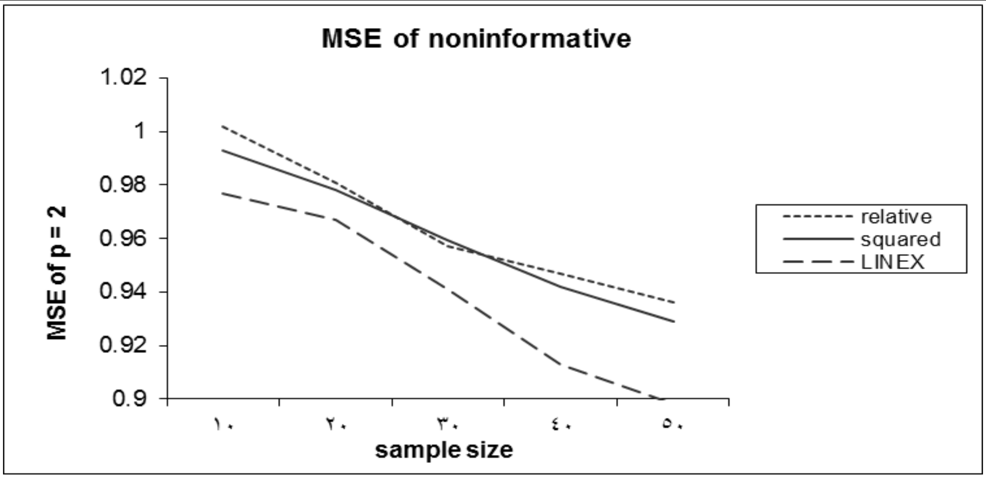
Fig(3) Bayesian estimation of the Exponentiated Frechet distribution when a,b,p are unknown



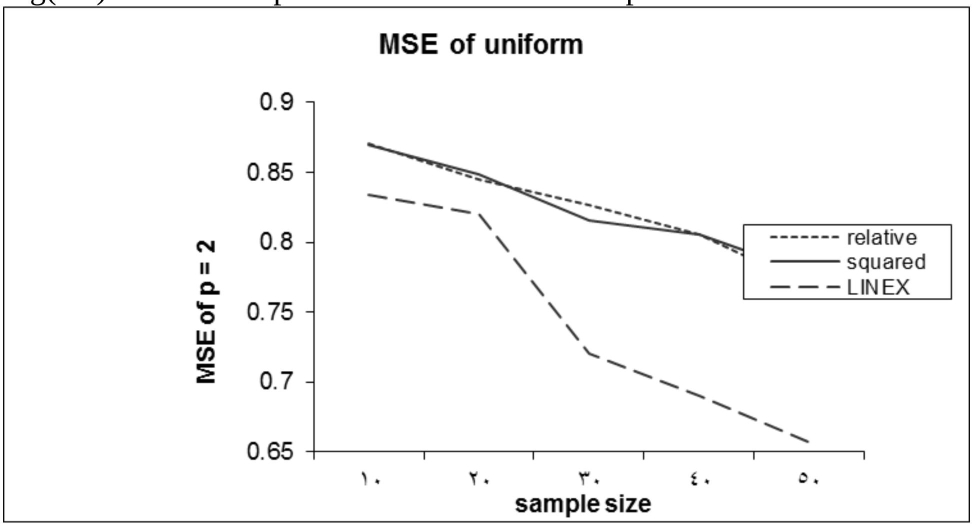
Fig(3.1)The MSE of p=3 under noninformative prior



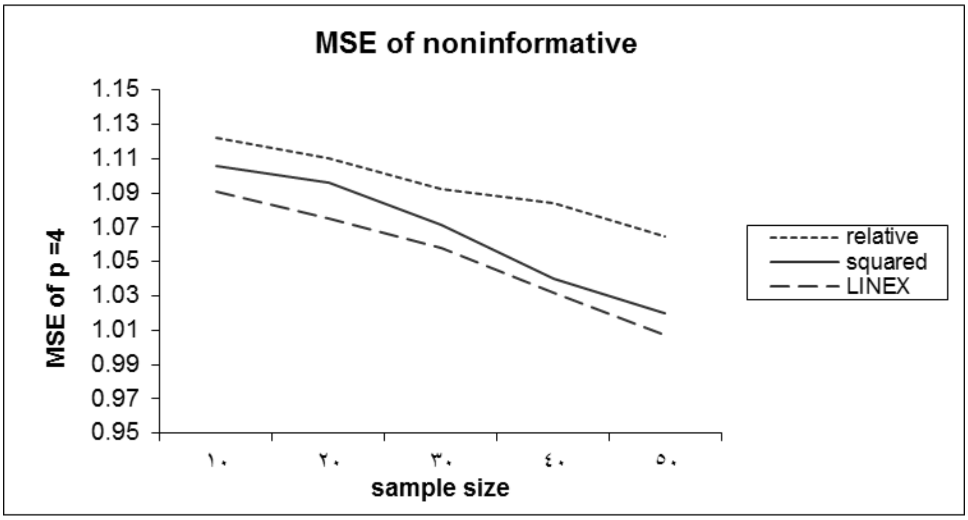
Fig(3.2)The MSE of p=3 under uniform prior



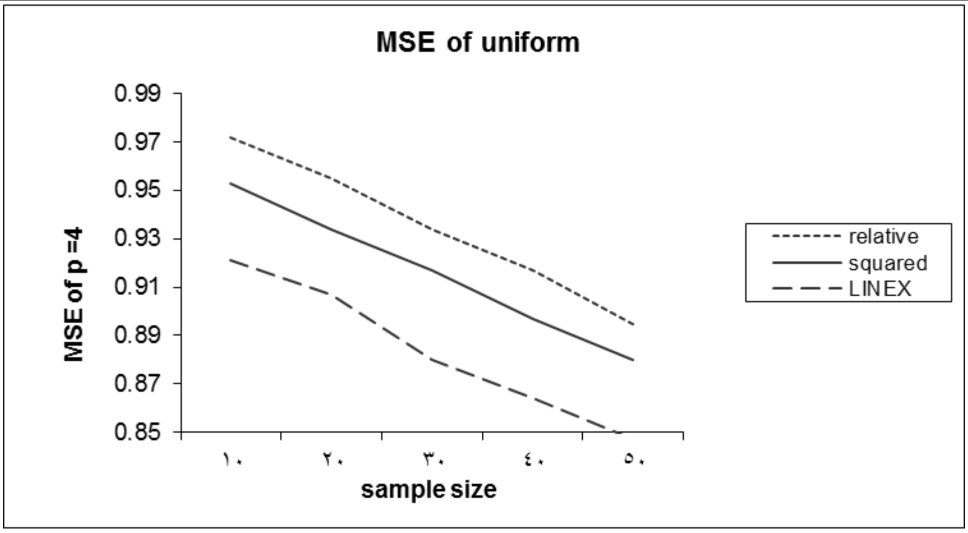
Fig(3.3)The MSE of p=2 under noninformative prior



Fig(3.4)The MSE of p=2 under uniform prior



Fig(3.5)The MSE of p=4 under noninformative prior



Fig(3.6)The MSE of p=4 under uniform prior