Some New Results Involving the NBUCA Class of Life Distributions

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Abstract

In this paper, some new characterization results of the NBUCA class of life distributions are obtained. A new approach is presented for testing constant versus NBUCA class of life distributions. The new test is much simpler to compute asymptotically normal and performs better than previous tests in terms of power and Pitman asymptotic efficiencies for several alternatives.

Keywords: Goodness of fit testing, Hypothesis testing, Asymptotic normality, Efficiency, Monte Carlo methods, NBUCA, NBU, NBUC.

1 Introduction

Aging is described by a non negative random variable with distribution function and survival function . For practicality, it is often assumed (but need not be) continuous with probability density function . Aging distributions are divided into many non-parametric classes according to the type of aging that take place. For example, we mention here rather than new better (worse) than used (NBU(NWU)), new better(worse) than used in the increasing convex order (NBUC(NWUC)). In this work, we focus our attention on the new better than used in the increasing convex average order aging class (NBUCA). First we review some common notions of stochastic orderings and aging notions that are considered in the paper (see, Shaked and Shanthikumar (2007) and Barlow and Proschan (1981)). Formally, if X and Y are two random variables with distributions F and G (survivals $\bar{F}$ and $\bar{G}$), respectively, then we say that X is smaller than Y in the

(a) Stochastic order sense (denoted by $X \leq_{st} Y$) if, and only, if

$$\bar{F}(x) \leq \bar{G}(x), \quad \text{for all } x;$$

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(b) increasing convex order (denoted by X $\leq_{icx}$ Y) if, and only, if

$$\int_x^{\infty} \bar{F}(u)du \leq \int_x^{\infty} \bar{G}(u)du, \quad \text{for all } x.$$

(c) increasing convex average order (denoted by X $\leq_{icxa}$ Y) if, and only, if

$$\int_0^{\infty}\int_x^{\infty} \bar{F}(u)dudx \leq \int_0^{\infty}\int_x^{\infty} \bar{G}(u)dudx, \quad \text{for all } x \geq 0.$$

Applications, properties and interpretations of the increasing convex average order in the statistical theory of reliability, and in economics can be found in Ahmad et al. (2006), Ahmad and Kayid (2006) and Al-Nachawati (2006).

On the other hand, in the context of lifetime distributions, some of the above orderings have been used to give characterizations and new definitions of aging classes. By aging, we mean the phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense, than a younger one (Bryson and Siddiqui (1969)). One of the most important approaches to the study of aging is based on the concept of the residual life. For any random variable X, let

$$X_t = [X - t \mid X > t], \quad t \in \{x\text{: } F(x)<1\},$$

denote a random variable whose distribution $F_t(x)$ is the same as the conditional distribution of X-t given that X>t and has survival function $\bar{F}_t(x) = \bar{F}(x+t)/\bar{F}(t)$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. We say that a non-negative random variable X , or its distribution F, is (Barlow and Proschan (1981) and Deshpande et al. (1986)).

(a) new better(worse) than used (denoted by $X \in NBU(NWU)$) if

$$\bar{F}(x+t) \leq (\geq) \bar{F}(x)\bar{F}(t), \quad \text{for all } x, t \geq 0;$$

(b) new better(worse) than used in the increasing convex order (denoted by $X \in NBUC(NWUC)$) if

$$\int_x^{\infty} \bar{F}(u+t)du \leq (\geq) \bar{F}(t)\int_x^{\infty} \bar{F}(u)du, \quad \text{for all } x, t \geq 0;$$

Recently, based on the increasing convex average order, Ahmad et al. (2006) introduced a new aging class of life distributions. Its definition is also recalled here .

Definition 1.1.
A non-negative random variable X is said to be new better(worse) than used in the increasing convex average order (denoted by $X \in NBUCA$) if, and only, if

$$\int_0^\infty \int_x^\infty \bar{F}(u+t)du \le \bar{F}(t)\int_0^\infty \int_x^\infty \bar{F}(u)du, \quad \text{for all } t \ge 0 \tag{1.2}$$

It is obvious that (1.2) is equivalent to

$$X_t \le_{icxa} X, \quad \text{for all } t \ge 0.$$

The relations between the above stochastic orderings and aging classes are as follows:

$$X \le_{st} Y \Rightarrow X \le_{icx} Y \Rightarrow X \le_{icxa} Y \tag{1.3}$$

And

$$NBU \subset NBUC \subset NBUCA$$

In the current investigation, we further develop the NBUCA class. In Section 2, we provide some new characterizations and preservation properties for the NBUCA aging notion. In Section 3, we provide a procedure based on the goodness of fit approach to test that X is exponential against that it is NBUCA and not exponential. This procedure is simpler to compute and has higher asymptotic Pitman relative efficiency for several alternatives, including three alternatives given by Hollander and Proschan (1975). The asymptotic normality of the proposed statistic is presented. Finally, in Section 4, the Pitman asymptotic efficacy, the power and the critical values of the proposed statistic are also calculated.

2 Characterizations results

In this section, we provide some new preservation and characterization results concerning the NBUCA class. The next theorem states that the NBUCA aging notion is preserved under increasing convex transformations

Theorem 2.1 .

Let X be non-negative random variable. If X is NBUCA then $h(X_t) \leq_{icxa} h(X)$, for all non-negative increasing convex function h .

Proof .

Let h be any non-negative function increasing convex, and suppose that X is NBUCA. We have to prove that

$$[h(X_s) \mid h(X_t) > s] \leq_{icxa} [h(X\ ) \mid h(X\ ) > s], \quad \text{for all } s < \min(h(u_{X_t}), h(u_X)). \quad (2.1)$$

From the assumption, it follows that $(X_t)_{h^{-1}(s)} \leq_{icxa} (X\ )_{h^{-1}(s)}$ or, equivlently,

$$[X_t \mid X_t > h^{-1}(\text{s}) \leq_{icxa} X \mid X > h^{-1}(\text{s})\ ], \quad \text{for each s.} \quad (2.2)$$

Here the inverse h^{-1} of h is taken to be the right continuous version of it defined by $h^{-1}(u) = \sup\{x : h(x) \leq u\}$ for $u \in \Re$. From the definition of h^{-1} and the continuity of h , it is easy to check that $x > h^{-1}(s)$ if, and only, if $h(x) > s$. Thus (2.2) can be rewritten as

$$[X_t \mid h\ (\text{X}_t) > \text{s}] \leq_{icxa} [X\ \mid h\ (\text{X}\ ) > \text{s}], \quad \text{for each s and t } \geq 0.$$

We get that

$$h([X_t \mid h\ (\text{X}_t) > \text{s}])\ \leq_{icxa}\ \text{h}([X\ \mid h\ (\text{X}\ ) > \text{s}]),$$

implying (2.1). This completes the proof.□

Suppose now that $X_1, X_2, \ldots$ be a sequence of independent and identical distributed (i.i.d) random variables and N be a positive integer-valued random variable which is independent of the X_i Put

$$X_{(1:N)} \equiv \min\{X_1, X_2, \ldots, X_N\}.$$

The random variables $X_{(1:N)}$ arise naturally in reliability theory as the lifetimes of series systems, with the random number N of identical components with lifetimes $X_1, X_2, \ldots, X_N$. In life-testing, if a random censoring is adopted, then the completely observed data constitute a sample of random size, say $X_1, X_2, \ldots, X_N$ where $N > 0$ is a random variable of integer value. In survival analysis, $X_{(1:N)}$ arises naturally as the minimal survival time of a transplant operation, where N of them are defective and hence may cause death .

To state and prove the closure of NBUCA class under series systems which are composed of a random number of i.i.d. components, we need the following preliminary results, which is due to Shaked and Shanthikumar (2007).

Lemma 2.1 .

Let the independent non-negative random variables $X_1, X_2, \ldots, X_n$, $Y_1, Y_2, \ldots, Y_n$ have the survival functions $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n$, $\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_n$ respectively. If, $X_i \leq_{icxa} Y_i$, i=1,2,...n, then

$$\min\{X_1, X_2, \ldots, X_n\} \leq_{icxa} \min\{Y_1, Y_2, \ldots, Y_n\}.$$

Lemma 2.2 .

Let X, Y, and N be random variables such that $[X \mid N = n] \leq_{icxa} [Y \mid N = n]$ for all n in the support of N. Then $X \leq_{icxa} Y$. That is, the increasing convex average is closed under mixtures .

Suppose now that $X_1, X_2, \ldots, X_N$ *and* $Y_1, Y_2, \ldots, Y_N$ have the survival functions $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_N$, and $\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_N$ respectively and N is independent of X_i's and Y_i's. Then by *Lemmas 2.1* and 2.2, $X_i \leq_{icxa} Y_i$ for i=1,2,... implies

$$\min\{X_1, X_2, \ldots, X_N\} \leq_{icxa} \min\{Y_1, Y_2, \ldots, Y_N\}. \tag{2.3}$$

We also observe that, given a set $X_1, X_2, \ldots, X_N$ of independent components and letting $\mathrm{T}_N = \tau(X_1, X_2, \ldots, X_N)$ be the random lifetime of coherent system with components $X_1, X_2, \ldots, X_N$ we have

$$[\mathrm{T}_N - t \mid \mathrm{T}_N > t] \leq_{st} \tau\{[X_1 - t \mid X_1 > t], \ldots, [X_N - t \mid X_N > t]\}. \tag{2.4}$$

Theorem 2.2.

Let $X_1, X_2, \ldots, X_N$ be a set of NBUCA independent components, with non-negative increasing convex survival functions and N be a positive integer-valued random variable. Then $X_{(1:N)} \in NBUCA$.

Proof .

By (2.4) and (1.3) and we have that

$$\left[X_{(1:N)} - t \mid X_{(1:N)} > t\right] \leq_{icxa} \min\left\{\left[X_1 - t \mid X_1 > t\right], \ldots, \left[X_N - t \mid X_N > t\right]\right\},$$

Let $\bar{F}_{i|t}$ be the survival function of $[X_i - t \mid X_i > t]$. Then $\bar{F}_{i|t}$ is non-negative increasing convex if $\bar{F}_i$ is non-negative increasing convex (see, Gao et al. (2002)). Now by the assumption and (2.3) we get

$$\min\left\{\left[X_1 - t \mid X_1 > t\right], \ldots, \left[X_N - t \mid X_N > t\right]\right\} \leq_{Lt} \min(X_1, X_2, \ldots, X_n).$$

Hence the result follows .

3 Testing against NBUCA alternatives

In the context of reliability and life testing, the hazard rate of a life distribution plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distribution and exhibits different monotonic behaviors. The concept of the ageless notion is equivalent to the phenomenon that age has no effect on the hazard rate. Thus the ageless property is equal to constant hazard rate, that is, the distribution is exponential. Hence testing non-parametric classes is done by testing exponentiality versus some kind of classes. This applies to many non-parametric classes such as NBU, NBUC, NBUT and IMIT, among many others. For a recent literature on testing the above classes as well as others we refer the readers to Ahmad (2001), Ahmad et al. (2001), Ahmad and Mugdadi (2004), Ahmad and Kayid (2004), Ahmad, Kayid and Pellerey (2005) and Ahmad, Kayid and Li (2005). Much of the earlier literature is cited in those papers where definitions, inter-relations and discussion of above classes are presented .

This section is divided into two main subsections. The first one is concerned with the construction of the proposed statistic as a goodness of fit, discussing its asymptotic normality and explaining how one can use it as application of testing of hypotheses. In the second subsection, the simulated upper $\alpha-$ percentile values for 90, 95 and 99 of the proposed statistic are presented and some applications are provided.

3.1 The goodness of fit test

The test presented here depends on a sample $X_1, X_2, \ldots, X_N$ from a population with distribution F. We wish to test the null hypothesis H□ : F is exponential with mean μ against H_1 : F is NBUCA and is not exponential. According to (1.2) we may use the following as a measure of departure from H□ in favor of H_1 :

$$\Delta(t) = \int_o^{\infty}\left[\bar{F}(t)\int_0^{\infty}\int_x^{\infty}\bar{F}(u)du dx - \int_0^{\infty}\upsilon(x+t)dx\right]dF_0(t) \tag{3.1}$$

where $\upsilon(x+t) = \int_x^{\infty}\bar{F}(u+t)du$.

If we denote by $dF_0(t) = e^{-t}du$ and by integration by parts for r.h.s. of (3.1), we get the following measure of departure,

$$\delta_{ca}(t) = \int_0^{\infty}\left\{\frac{\mu_2}{2}\bar{F}(t) - \int_0^{\infty}\upsilon(x+t)dx\right\}dF_0(t) \tag{3.2}$$

where $\frac{\mu_2}{2} = \int_0^{\infty}x\bar{F}(x)dx$

Note that : $H_0 : \delta_{ca} = 0$ versus $H_1 : \delta_{ca} > 0$.

Lemma 3.1 .

Let $T \geq 0$ be a random variable with distribution F (.), then

$$\delta_{ca}(t) = E\left[\frac{\mu_2}{2}(1-e^{-T}) - \frac{T^2}{2} + T - (1-e^{-T})\right]. \tag{3.3}$$

Proof .

Note that δ_{ca} given in (3.2) can be written as

$$\delta_{ca}(t) = \int_0^{\infty}\left\{\frac{\mu_2}{2}\bar{F}(t) - \int_0^{\infty}\upsilon(x+t)dx\right\}e^{-t}dt$$

$$= E\left[\int_0^{\infty}\frac{\mu_2}{2}I(T>t)e^{-t}dt\right] - \int_0^{\infty}\int_0^{\infty}\left[x\bar{F}(x+t)\right]dxe^{-t}dt,$$

let $x+t=y \Rightarrow x = y-t$, $x \geq 0$, then

$$\bar{F}_1(x)=e^{-x^{\theta}}, \quad x\geq 0, \quad \theta\geq 0;$$

(ii) Linear Failure Rate Family (F_2) :

$$\bar{F}_2(x)=e^{-x-\frac{\theta}{2}x^2}, \quad x\geq 0, \quad \theta\geq 0;$$

(iii) Gamma Family (F_3) :

$$\bar{F}_3(x)=[\Gamma(\theta)]^{-1}\int_x^{\infty} e^{-u}u^{\theta-1}du, \quad x\geq 0, \quad \theta\geq 0.$$

Note that the Pitman asymptotic efficacy (PAE) is defined by:

$$PAE(\delta_{ca})=\frac{d}{d\theta}\delta ca\,|_{\theta\to\theta_0}\,/\,\sigma_0.$$

Carrying out the efficiency calculations for the above three alternatives, namely Weibull, linear failure rate and Makeham we get 0.714, 0.197 and 0.714.

4.1 The power of the proposed test

The power of the proposed test at a significance level α with respect to the alternatives F_1, F_2 and F_3 is calculated based on simulation data. In such simulation, 25000 samples were generated with sizes n=10, 20 and 30 from the alternatives. Table (2) shows the power of test at different values of θ and the significance level α=0.05.

Table (2)

The power of the proposed test $\hat{\delta}_{ca}$

n	θ	Weibull (F_1)	LFR (F_2)	Gamma (F_3)
10	2	0.30700	0.00100	0.34400
	3	0.68700	0.00000	0.55600
	4	0.91100	0.00000	0.47900
20	2	0.83800	0.02100	0.57400
	3	1.00000	0.00200	0.89200
	4	1.00000	0.00000	0.93800
30	2	0.97500	0.04100	0.67900
	3	1.00000	0.02300	0.97700
	4	1.00000	0.00200	0.99700

From the above table, it is noted that the power of the test increases by increasing the values of the parameter θ with respect to the alternatives F_1 and F_3 and the sample size n as it was expected

4.2 Numerical example

Consider the following data of Bryson and Siddiqui (1969), which represent the survival times of 43 patients suffering from chronic granulocytic leukemia and the ordered life times (in days) are :

.13, 14, 19, 19, 20, 21, 23, 23, 25, 26, 26, 27, 27, 31, 32, 34, 34, 37, 38, 40, 46, 50, 53, 54, 57, 58, 59, 60, 65, 65, 66, 70, 85, 90, 98, 102, 103, 110, 118, 124, 130, 136, 138.

Calculating $\sqrt{n}\ \hat{\delta}_{ca} / \sigma_0$ we get 0.017133, which is smaller than Z_α for any α. This value leads to the acceptance of H_0 agreeing with conclusion of Bryson and Siddiqui (1969).

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