GENERALIZED MATRIX VARIATE GAMMA DISTRIBUTION: SOME PROPERTIES AND RELATED DISTRIBUTIONS

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ABSTRACT

A new type of generalized matrix variate gamma distribution is defined. The proposed distribution is the matrix variate analog of Agarwal and Kalla (1996) scalar generalized gamma distribution. Some of the well known matrix variate distributions are shown to be its particular cases. The main statistical properties of the proposed generalized matrix variate gamma distribution are work out. Finally, some probability distributions connected with the generalized matrix variate gamma distribution are introduced.

§ 1. INTRODUCTION

In recent years many generalizations of gamma and Weibull distributions are proposed notably by Bradeley (1988). Srivastava (1989), Lee and Gross (1991), and Bondesson (1992). These generalized distributions are mainly introduced in order to extend the scope of applications of ordinary gamma and Weibull distributions.

Kobayashi (1991) has introduced a new type of generalized gamma function as

$$\Gamma(m,n,r) = \int_0^{\infty} e^{-x} \quad x^{m-1} \ (x+n)^{-r} \quad dx \qquad \ldots(1\text{-}1)$$

for a positive integer r. Here m and n are parameters of the function. This function occurs in many problems of differaction theory [Kobayashi (1991)].

In order to define a new type of generalization of the gamma distribution Agarwal and Kalla (1996) considered a slightly modified form of the generalized gamma function as

$$\int_0^\infty e^{-\alpha x} \quad x^{m-1}(x+n)^{-\lambda}\,dx \qquad\qquad \alpha, m, n > 0$$

$$= \alpha^{\lambda\text{-}m} \quad \Gamma(m,\alpha n,\lambda) \qquad \ldots(1\text{-}2)$$

Moreover, a random variable X follows a new type of generalized gamma distribution with four parameters if its probability density function (p.d.f) is given [Agarwal and Kalla (1996)] by:

$$f(x,m,n,\alpha,\lambda)$$

$$= \frac{\alpha^{m-\lambda}}{\Gamma(m,\alpha n,\lambda)} e^{-\alpha x} x^{m-1}(x+n)^{-\lambda} \qquad x > 0 \qquad \ldots(1\text{-}3)$$

where m is the shape parameter, α is the scale parameter, n is the displacement paraemter, and λ is the parameter of intensity of the effect of the corresponding displacement parameter.

In this paper, we define a new generalized matrix variate gamma density. The proposed distribution is the matrix variate analog of Agarwal and Kalla (1996) scalar generalized gamma distribution defined in (1-3) above.

In section (2), the basic notations, definitions, and theorems of functions of matrix arguments that are needed in the rest of the paper will be presented. In section (3), we have defined a new type of generalized matrix variate gamma distribution. In section (4), the main statistical properties of the proposed generalized matrix variate gamma distribution are work out. Finally, some probability distributions connected with the generalized matrix variate gamma distribution are introduced in section (5).

§ 2. FUNCTIONS OF MATRIX ARGUMENTS: SOME BASIC PRELIMINARY RESULTS

First we will start with the discussion of scalar function of matrix arguments. All the matrices appearing in this paper are assumed to be symmetric positive definite and having only real (not complex) elements unless otherwise specified.

Let $\mathbf{A} = [a_{ij}]$ be a pxp real symmetric matrix. Due to symmetry all the p^2 elements are not distinct only a maximum of $1+2+\ldots+p = \frac{p(p+1)}{2}$ elements are distinct.

(2-1) Notations

$\mathbf{A} > \mathbf{0}$: the matrix $\mathbf{A} = [a_{oj}]$ is positive definite
$\mathbf{A} \geq \mathbf{0}$: the matrix $\mathbf{A}$ is positive semi-definite
$\mathbf{A} < \mathbf{0}$: the matrix $\mathbf{A}$ is negative definite
$\mathbf{A} \leq \mathbf{0}$: the matrix $\mathbf{A}$ is negative semi-definite
$\mathbf{0} < \mathbf{A} < \mathbf{B}$: the matrices **A,B, B-A** are positive definite
$|\mathbf{A}| = \det \mathbf{A}$: determinant of **A**
$\underline{tr}\ \mathbf{A} = \underline{tr}\ (\mathbf{A}) =$ trace of $\mathbf{A} = a_{11} + a_{22} + \ldots + a_{pp}$

When dealing with functions of matrices it is often necessary to transform a pxp matrix **X** to the pxp matrix $\mathbf{Y} = \mathbf{F}(\mathbf{X})$ when **F** may or may not be linear.

The transformation $\mathbf{Y} = \mathbf{F}(\mathbf{X})$, $\mathbf{X} = \mathbf{X}'$, $\mathbf{Y} = \mathbf{Y}'$, will be treated as a transformation of the p(p+1)/2 functionally independent scalar variables in **X** to p(p+1)/2 funcationally indepedent scalar variables in **Y**. If **X** and **Y** are not symmetric then it is a transformation of p^2 variable to P^2 variables.

If a pxp symmetric matrix **X** is transformed to **Y** then the Jacobian of the transformation is defined as

$$\mathbf{J}(\mathbf{X} \rightarrow \mathbf{Y}) = \left|\frac{\partial \mathbf{X}}{\partial \mathbf{Y}}\right|_{+}$$

where (+) indicates the absolute value of the determinant, and $\frac{\partial \mathbf{X}}{\partial \mathbf{Y}}$ is a square matrix of order p(p+1)/2 (when **X** and **Y** are symmetric) and its i-th row j-th column element here is the partial derivative of the i-th row element in **X** with respect to j-th column element in **Y**.

The Jacobians of certain transformations which are needed in the subsequent sections are given in Deemer and Olkin (1951), Olkin (1953), Rogers (1980), and Magnns and Neudecker (1988).

(2-2) Integration

Some useful integrals which are needed in the following sections are now given.

Definition (1): The matrix variate gamma function

The matrix variate gamma function, denoted by $\Gamma_p(\alpha)$, is defined as

$$\Gamma_p(\alpha) = \int_{A>0} e^{-tr(A)} |\mathbf{A}|^{\alpha - ½(p+1)} dA$$

$$Re(\alpha) > ½(p-1) \qquad \ldots(2\text{-}1)$$

Where the integral is over the space of pxp symmetric positive definite matrices.

The matrix variate gamma function $\Gamma_p(\alpha)$ can be expressed as product of ordinary scalar gamma functions given in the following Lemma.

Lemma (1)

For $Re(\alpha) > ½(p-1)$,

$$\Gamma_p(\alpha) = \pi^{¼p(p-1)} \Gamma(\alpha)\, \Gamma(\alpha-½)\, \Gamma(\alpha-1) \ldots \Gamma(\alpha- \frac{p-1}{2})$$

$$= \pi^{¼p(p-1)} \prod_{i=1}^{p} \Gamma[\alpha-½\,(i-1)] \qquad \ldots(2\text{-}2)$$

Proof: See Muirhead [(1982) p. 61]

A particular Laplace transform which is quite useful is

$$\int_{A>0} e^{-tr(ZA)} |\mathbf{A}|^{\alpha - ½(p+1)} dA$$

$$= |\mathbf{Z}|^{-\alpha} \Gamma_p(\alpha) \qquad \ldots(2\text{-}3)$$

Herz (1955) proved that the above integral is absolutely convergent for $Re(\mathbf{z}) > 0$, and $Re(\alpha) > \frac{1}{2}(p-1)$. Hence, for $Re(\mathbf{z}) > 0$, substituting $B = \mathbf{Z}^{\frac{1}{2}} A \mathbf{Z}^{\frac{1}{2}}$ with the Jacobian $J(\mathbf{A} \to \mathbf{B}) = |\mathbf{Z}|^{-\frac{1}{2}(p+1)}$ in the above integral we get

$$\int_{A>0} e^{-tr(\mathbf{ZA})} |\mathbf{A}|^{\alpha - \frac{1}{2}(p+1)} dA$$

$$= |\mathbf{Z}|^{-\alpha} \int_{\mathbf{B}>0} e^{-tr(\mathbf{B})} |\mathbf{B}|^{\alpha - \frac{1}{2}(p+1)} d\mathbf{B}$$

$$= |\mathbf{Z}|^{-\alpha} \Gamma_p(\alpha)$$

This proves (2-3).

Definition (2): The matrix variate type-1 beta function

The matrix variate type-1 beta function, denoted by $\beta_p(a,b)$, is defined by

$$\beta_p(a,b) = \int_{0<A<I_p} |\mathbf{A}|^{a - \frac{1}{2}(p+1)} |\mathbf{I}-\mathbf{A}|^{b - \frac{1}{2}(p+1)} d\mathbf{A}$$

$$Re(a) > \tfrac{1}{2}(p-1),\ Re(b) > \tfrac{1}{2}(p-1) \qquad \ldots(2\text{-}4)$$

Substituting $\mathbf{A} = (\mathbf{I}_p+\mathbf{B})^{-1}$ in (2-4) with Jacobian $J(\mathbf{A} \to \mathbf{B})\ |\mathbf{I}_p+\mathbf{B}|^{-(p+1)}$ we get the following:

Definition (3): The matrix variate type-2 beta function

For $Re(a) > \frac{1}{2}(p-1)$ and $Re(b) > \frac{1}{2}(p-1)$

$$\beta_p(a,b) = \int_{\mathbf{B}>0} |B|^{b - \frac{1}{2}(p+1)} |I_p+\mathbf{B}|^{-(a+b)} d\mathbf{B} \qquad \ldots(2\text{-}5)$$

The matrix variate beta function $\beta_p(a,b)$ can be expressed in terms of matrix variate gamma functions.

Lemma (2):

For $Re(a) > \frac{1}{2}(p-1)$ and $Re(b) > \frac{1}{2}(p-1)$,

$$\beta_p(a,b) = \frac{\Gamma_p(a)\,\Gamma_p(a)}{\Gamma_p(b,a)}$$

$$= \beta_p(b,a) \qquad \ldots(2\text{-}6)$$

(2-3) Hypergeometric Functions of Matrix Arguments

Distributional results of random matrices are often derived in terms of hypergeometric functions of matrix arguments. Some useful results which are needed in the sequel are now given.

Using the Laplace and inverse Laplace transforms for the matrix variate case Herz (1955) defined a hypergeometric function of a matrix argument by using the following pair of equations:

$$_{r+1}F_s(a_1,\dots,a_r,c;\, b_1,\dots,b_s;-\mathbf{Z}^{-1})\,|\mathbf{Z}|^{-c}$$

$$= \frac{1}{\Gamma_p(c)} \int_{\Lambda>0} e^{-\mathrm{tr}(\mathbf{Z}\Lambda)}\, {}_rF_s(a_1,\dots,a_r;\, b_1,\dots,b_s;-\Lambda)\,|\Lambda|^{c-\frac{1}{2}(p+1)}\, d\Lambda \qquad \dots(2\text{-}7)$$

and

$$_rF_{s+1}(a_1,\dots,a_r;\, b_1,\dots,b_s,c;-\Lambda)\,|\Lambda|^{c-\frac{1}{2}(p+1)}$$

$$= \frac{\Gamma_p(c)(2)^{\frac{1}{2}p(p-1)}}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{\mathrm{Re}(z)=X_0>0} e^{-\mathrm{tr}(\Lambda\mathbf{Z})}\, {}_rF_s(a_1,\dots,a_r;\, b_1,\dots,b_s;-z^{-1})\,|\mathbf{Z}|^{-c}\, d\mathbf{Z} \qquad \dots(2\text{-}8)$$

Note that equations (2-7) and (2-8) do not give the hypergeometric functions explicitly except for special cases. But these equations enable us to study the properties. The conditions to be satisfied are that $\mathrm{Re}(\mathbf{Z}) > 0$, $\Lambda > \mathbf{0}$, $\mathrm{Re}(c) > \frac{1}{2}(p-1)$, $s \geq r$ or $r = s+1$ and $\|\mathbf{Z}\| < 1$ where $\|\mathbf{Z}\|$ denotes a norm of $\mathbf{Z}$. The parameters $b_1,\dots,b_s$ are such that none of $b_j-\frac{1}{2}(k-1)$, $j = 1,\dots,s$, $k = 1,\dots,p$ is a negative integer or zero. If any of the $b_j-\frac{1}{2}(k-1)$, $j = 1,\dots,s$, $k = 1,\dots,p$ is a negative integer or zero then there should be an a_j, $j=1,\dots,r$ such that $(a_j-\frac{k-1}{2})_m = 0$ first before any $(b_\ell-\frac{k-1}{2})_m = 0$ for $j=1,\dots,r$, $k=1,\dots,p$, $\ell = 1,\dots,s$ and for exmple $(a)_m = a(a+1)\dots(a+m-1)$, $(a)_0 = 1$, $a \neq 0$.

By using (2-7) and (2-8) one can extend most of the integrals involving hypergeometric functions in the scalar case to the corresponding matrix case. Some of these results which are useful in our derivations are now given.

(i) $\quad {}_0F_0(\mathbf{Z}) = e^{\mathrm{tr}(\mathbf{Z})}$ $\qquad \dots(2\text{-}9)$

(ii) $${}_1F_0(c;\ ;Z) = |I_p - Z|^{-c} \qquad \|\mathbf{Z}\| < 1 \qquad \text{...(2-10)}$$

(iii) Let R(pxp) be a symmetric matrix, then

$$\int_{0<S<I_p} |\mathbf{S}|^{a-\frac{1}{2}(p+1)} |\mathbf{I}_p-\mathbf{S}|^{b-\frac{1}{2}(p+1)} {}_mF_n(a_1,\dots,a_m;b_1,\dots,b_n,\mathbf{RS})d\mathbf{S}$$

$$= \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)} {}_{m+1}F_{n+1}(a_1,\dots,a_m;a;\ b_1,\dots,b_n,a+b;R) \qquad \text{...(2-11)}$$

(iv) For Re(α) > ½(p-1), Re(β) > ½(p-1) and Re(β-α) > ½(p-1) and symmetric R(pxp)

$${}_1F_1(\alpha;\beta;R)$$

$$\frac{\Gamma_p(\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta-\alpha)} \int_{0<S<I_p} |\mathbf{S}|^{\alpha-\frac{1}{2}(p+1)} |\mathbf{I}_p-\mathbf{S}|^{\beta-\alpha-\frac{1}{2}(p+1)} e^{\underline{tr}(\mathbf{RS})} d\mathbf{S} \qquad \text{...(2-12)}$$

(v) For Re(α) > ½(p-1), Re(γ-α) > ½(p-1) and symmetric **R**(pxp)

$${}_2F_1(\alpha;\beta;\gamma;\mathbf{R}) = \frac{\Gamma_p(\gamma)}{\Gamma_p(\alpha)\Gamma_p(\gamma-\alpha)}$$

$$\int_{0<S<I_p} |\mathbf{S}|^{\alpha-\frac{1}{2}(p+1)} |\mathbf{I}_p-\mathbf{S}|^{\gamma-\alpha-\frac{1}{2}(p+1)} |\mathbf{I}_p-\mathbf{S}|^{-\beta} d\mathbf{S} \qquad \text{...(2-13)}$$

(vi) For Re(γ) > ½(p-1) and Re(γ-α-β) > ½(p-1)

$${}_2F_1(\alpha;\beta;\gamma;\mathbf{I}_p) = \frac{\Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)}{\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\beta)} \qquad \text{...(2-14)}$$

(vii) The hypergeometric functions with matrix arguments ${}_1F_1$ and ${}_2F_1$ as given in (2-33) and (2-34) respectively, satisfy the following relations [Herz, 1955].

$${}_1F_1(\alpha;\gamma;\mathbf{S}) = e^{\underline{tr}(\mathbf{S})}\ {}_1F_1(\gamma-\alpha;\gamma;-\mathbf{S}) \qquad \text{...(2-15)}$$

$$ {}_2F_1(\alpha;\beta;\gamma;\mathbf{S}) = |\mathbf{I}_p-\mathbf{S}|^{-\beta}\ {}_2F_1(\gamma-\alpha;\beta;\gamma;-\mathbf{S}(\mathbf{I}_p-\mathbf{S})^{-1}) \qquad ...(2\text{-}16)$$

$$ = |\mathbf{I}_p-\mathbf{S}|^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma-\alpha;\gamma-\beta;\gamma;\mathbf{S}) \qquad ...(2\text{-}17)$$

(viii) There is yet another type of confluent hypergeometric function, Ψ, of matrix argument defined as [Muirhead, 1970]

$$\Psi(a;c;R)$$

$$= \frac{1}{\Gamma_p(a)} \int_{S>0} e^{-tr(\mathbf{RS})} |\mathbf{S}|^{a-½(p+1)} |\mathbf{I}_p+\mathbf{S}|^{c-a-½(p+1)}\ \mathbf{dS}$$

$$= \frac{1}{\Gamma_p(a)|\mathbf{R}|^{c-\frac{1}{2}(p+1)}} \int_{S>0} e^{-tr(\mathbf{S})} |\mathbf{S}|^{a-½(p+1)} |\mathbf{R}+\mathbf{S}|^{c-a-½(p+1)}\ \mathbf{dS} \qquad ...(2\text{-}18)$$

where R(pxp) is a symmetric matrix, Re($\mathbf{R}$) > 0, and Re(a) > ½(p-1).

Moreover, Subrahmaniam (1973) proved the following results

(IX) For Re(γ) > ½(p-1) and Re($\theta+\alpha-\gamma$) > ½(p-1)

$$\int_{A>0} |\mathbf{A}|^{\gamma-½(p+1)} |\mathbf{I}_p-\mathbf{A}|^{-\theta}\ {}_2F_1(\alpha,\beta;\gamma;-\mathbf{A}|\ \mathbf{dA}$$

$$= \frac{\Gamma_p(\gamma)\Gamma_p(\alpha+\theta-\gamma)\Gamma_p(\beta+\theta-\gamma)}{\Gamma_p(\theta)\Gamma_p(\alpha+\beta+\theta-\gamma)} \qquad ...(2\text{-}19)$$

(X) For Re(α) > ½(p-1) and Re(b) > ½(p-1), and $0 < \mathbf{B} < I_p$

$$\int_{0<A<I_p} |\mathbf{A}|^{a-½(p+1)} |\mathbf{I}_p-\mathbf{A}|^{b-½(p+1)}\ {}_2F_1(\alpha,\beta;\gamma;-\mathbf{AB}|\ \mathbf{dA}$$

$$= \beta_p(a,b)\ {}_3F_2(a,\alpha,\beta;a+b,\gamma;\mathbf{B}) \qquad ...(2\text{-}20)$$

(XI) For $\mathbf{B}$(pxp) symmetric positive definite matrix, Re(-$\mathbf{BC}$) < I_p, and Re(a) > ½(p-1),

$$\int_{0<A<B} |\mathbf{A}|^{a-\frac{1}{2}(p+1)} |\mathbf{I}_p+\mathbf{CA}|^{-a-b} \, d\mathbf{A}$$

$$= \beta_p(a,\tfrac{1}{2}(p+1)) \, |\mathbf{B}|^a \, {}_2F_1(a,a+b;a+\tfrac{1}{2}(p+1);-\mathbf{BC}) \quad \ldots(2\text{-}21)$$

(XII) For $\mathbf{B}(p\mathrm{x}p)$ symmetric positive definite matrix, $\mathrm{Re}(-(\mathbf{BC})^{-1}) < I_p$, and $\mathrm{Re}(b) > \frac{1}{2}(p-1)$,

$$\int_{A>B} |\mathbf{A}|^{a-\frac{1}{2}(p+1)} |\mathbf{I}_p+\mathbf{CA}|^{-a-b} \, d\mathbf{A}$$

$$= \beta_p(b,\tfrac{1}{2}(p+1)) \, |\mathbf{B}|^{-b} \, |\mathbf{C}|^{-a-b}$$

$$ {}_2F_1(a,a+b;b+\tfrac{1}{2}(p+1);-(\mathbf{BC})^{-1}) \quad \ldots(2\text{-}22)$$

(XIII) For $\mathrm{Re}(I_p-\mathbf{B}^{-1}) < I_p$, $\mathrm{Re}(b+c-a) > \frac{1}{2}(p-1)$, and $\mathrm{Re}(a) > \frac{1}{2}(p-1)$

$$\int_{A>0} |\mathbf{A}|^{a-\frac{1}{2}(p+1)} |\mathbf{I}_p+\mathbf{A}|^{-b} \, |\mathbf{I}_p+\mathbf{BA}|^{-c} \, d\mathbf{A}$$

$$= \beta_p(a,b+c-a) \, |\mathbf{B}|^{-c} \, {}_2F_1(b+c-a,c;b+c;I_p-\mathbf{B}^{-1}) \quad \ldots(2\text{-}23)$$

(XIV) $$\int_{Y>0} e^{-\mathrm{tr}(\mathbf{XY})} |\mathbf{Y}|^{b-\frac{1}{2}(p+1)} \, {}_2F_1(a,a-c+\tfrac{1}{2}(p+1);b;-\mathbf{Y}) \, d\mathbf{Y}$$

$$= \Gamma_p(b) \, |\mathbf{X}|^{a-b} \, \Psi(a;c;x) \quad \ldots(2\text{-}24)$$

(XV) For $\mathrm{Re}(\mathbf{X}) > 0$, $\mathrm{Re}(b-c) > -1$, and $\mathrm{Re}(a) > \frac{1}{2}(p-1)$

$$\int_{Y>0} e^{-\mathrm{tr}(\mathbf{XY})} |\mathbf{Y}|^{b-\frac{1}{2}(p+1)} \, \Psi(a;c;\mathbf{Y}) \, d\mathbf{Y}$$

$$= \frac{\Gamma_p(b)\Gamma_p(b-c+\frac{1}{2}(p+1))}{\Gamma_p(a+b-c+\frac{1}{2}(p+1))} \, {}_2F_1(b,b-c+\tfrac{1}{2}(p+1);a+b-c+\tfrac{1}{2}(p+1);I_p-\mathbf{X})$$

$$\ldots(2\text{-}25)$$

(XVI) For $Re(\mathbf{AX}^{-1}) < I_p$

$$\int_{\mathbf{Y}>0} e^{-tr(\mathbf{XY})} |\mathbf{Y}|^{b-\frac{1}{2}(p+1)} {}_1F_1(a;c;\mathbf{AY})\, d\mathbf{Y}$$

$$= \Gamma_p(b)\ |\mathbf{X}|^{-b}\ {}_2F_1(a,b;c;\mathbf{AX}^{-1}) \qquad \ldots(2\text{-}26)$$

§3. GENERALIZED MATRIX VARIATE GAMMA DISTRIBUTION

In order to define a new type of matrix variate gamma distribution, we first introduce the "Generalized Matrix Variate Gamma Function" which is the matrix analog of Kobayashi (1991) scalar generalized gamma function defined in (1-1) above.

Definition (4): The Generalized Matrix Variate Gamma Function

The generalized matrix variate gamma function, denoted by $\Gamma_p(\alpha,\lambda,R)$, is define by

$$\Gamma_p(\alpha,\lambda,R) = \int_{\mathbf{X}>0} e^{-tr(X)} |X|^{\alpha-\frac{1}{2}(p+1)} |R+X|^{-\lambda}\, d\mathbf{X} \qquad \ldots(3\text{-}1)$$

where $\mathbf{R}(pxp)$ is a symmetric matrix, $Re(\mathbf{R}) > 0$, $Re(\alpha) > \frac{1}{2}(p-1)$, $Re(\lambda)>0$ and the integral is over the space of pxp symmetric positive definite matrices.

Note that the determinant

$$\begin{aligned} |\mathbf{R}+\mathbf{X}| &= |\mathbf{R}|\ |\mathbf{I}_p + \mathbf{R}^{-1}\mathbf{X}| \\ &= |\mathbf{R}|\ |\mathbf{I}_p + \mathbf{X}\,\mathbf{R}^{-1}| \\ &= |\mathbf{R}|\ |\mathbf{I} + \mathbf{R}^{-\frac{1}{2}}\,\mathbf{X}\,\mathbf{R}^{-\frac{1}{2}}| \end{aligned} \qquad \ldots(3\text{-}2)$$

where $\mathbf{R}^{\frac{1}{2}}$ is the symmetric square root of $\mathbf{R} = \mathbf{R}' > 0$. Change $\mathbf{X}$ to $\mathbf{Y} = \mathbf{R}^{-\frac{1}{2}}\mathbf{X}\,\mathbf{R}^{-\frac{1}{2}}$ for fixed R. Then

$$\mathbf{X} = \mathbf{R}^{\frac{1}{2}}\,\mathbf{Y}\,\mathbf{R}^{\frac{1}{2}} \qquad \ldots(3\text{-}3)$$

and the Jacobian of the transformation is

$$J(\mathbf{X} \to \mathbf{Y}) = |\mathbf{R}|^{\frac{1}{2}(p+1)} \qquad \ldots(3\text{-}4)$$

Using (3-2),(3-3), and (3-4) in (3-1) we get

Defintion (5):

The generalized matrix variate gamma function $\Gamma_p(\alpha,\lambda,R)$ can be defined as

$$\Gamma_p(\alpha,\lambda,R) = |\mathbf{R}|^{\alpha-\lambda} \int_{Y>0} e^{-tr(RY)} |\mathbf{Y}|^{\alpha-\frac{1}{2}(p+1)} |\mathbf{I}+\mathbf{Y}|^{-\lambda} \, \mathbf{dY} \qquad \ldots(3\text{-}5)$$

It is interesting to note that the generalized matrix variate gamma function as defined in (3-5) can be represented in temrs of the confluent hypergeometric function with matrix argument of the second kind $\psi(a;c;R)$ (2-18). Therefore, we have

Definition (6):

The generalized matrix variate gamma function $\Gamma_p(\alpha,\lambda,R)$ can be defined as

$$\Gamma_p(\alpha,\lambda,R) = \Gamma_p(\alpha) \, |\mathbf{R}|^{\alpha-\lambda} \, \psi(\alpha;\alpha-\lambda+\tfrac{1}{2}(p+1);\mathbf{R}) \qquad \ldots(3\text{-}6)$$

Now, we are completely ready to define a new type of generalized matrix variate gamma distribution

Definition (7):

A pxp random symmetric positive definite matrix **X** is said to have a "Generalized matrix Variate Gamma Distribution", denoted as $GG_p(\alpha,\lambda,\mathbf{R})$, if its p.d.f is given by

$$\frac{|\mathbf{R}|^{\alpha-\lambda}}{\Gamma_p(\alpha,\lambda,\mathbf{R})} \, e^{-tr(\mathbf{RX})} |\mathbf{X}|^{\alpha-\frac{1}{2}(p+1)} |I_p+X|^{-\lambda} \quad \mathbf{X} > 0 \qquad \ldots(3\text{-}7)$$

where $R(pxp) > 0$ and $\alpha > \frac{1}{2}(p-1)$.

Particular Case : $\lambda = 0$

In such a case it is observed-from (3-1),(3-5) and (3-7)-that

$$\Gamma_p(\alpha,0,R) = \int_{X>0} e^{-tr(X)}\, X^{\alpha-\frac{1}{2}(p+1)} \quad dY$$

$$= \Gamma_p(\alpha) \qquad \text{Re}(\alpha) > \tfrac{1}{2}(p-1) \qquad \ldots(3\text{-}8)$$

which is the ordinary matrix variate gamma function [See for instance, Mathai (1993), p. 159].

$$GG_p(\alpha,0,\mathbf{R}),= \frac{|\mathbf{R}|^{\alpha}}{\Gamma_p(\alpha,0,\mathbf{R})}\; e^{-tr(\mathbf{RX})}\, |\mathbf{X}|^{\alpha-\frac{1}{2}(p+1)}$$

$$= \frac{|\mathbf{R}|^{\alpha}}{\Gamma_p(\alpha)}\; e^{-tr(\mathbf{RX})}\, |\mathbf{X}|^{\alpha-\frac{1}{2}(p+1)}$$

$$\mathbf{X} > 0,\ \mathbf{R} > 0,\ \text{Re}(\alpha) > \tfrac{1}{2}(p-1) \qquad \ldots(3\text{-}9)$$

Which is the ordinary matrix variate gamma density $G_p(\alpha,R)$ [Mathai (1993)-p. 160].

§ 4. STATISTICAL PROPERTIES OF GENERALIZED MATRIX VARIATE GAMMA DISTRIBUTION

In this section, we study some statistical properties of the random matrices distributed as generalized matrix variate gamma distribution.

Theorem (1): Laplace transform and moment generating function

If **X** is a pxp real symmetric positive definitie matrix having a generalized matrix variate gamma density (3-7), then the Laplace transformation of **X** is

$$L_x(\mathbf{T}) = E[e^{-tr(\mathbf{TX})}]$$

$$= \frac{\Gamma_p(\alpha,\lambda,\mathbf{R}+\mathbf{T})}{\Gamma_p(\alpha,\lambda,\mathbf{R})}\; |I_p + \mathbf{TR}^{-1}|^{\lambda-\alpha} \qquad \ldots(4\text{-}1)$$

and the moment generating function is

$$M_x(\mathbf{T}) = E[e^{tr(\mathbf{TX})}] = L_X(-\mathbf{T})$$

$$= \frac{\Gamma_p(\alpha,\lambda,\mathbf{R}-\mathbf{T})}{\Gamma_p(\alpha,\lambda,\mathbf{R})}\; |I_p - \mathbf{TR}^{-1}|^{\lambda-\alpha} \qquad \ldots(4\text{-}2)$$

where **R** is assumed to be symmetric positive definite, T is a real symmetric parameter matrix and $I_p+TR^{-1} > 0$.

Proof:

By definition the Laplace transform is given by

$$L_x(T) = E[e^{-tr(TX)}]$$

$$= \frac{|\mathbf{R}|^{\alpha-\lambda}}{\Gamma_p(\alpha,\lambda,\mathbf{R})} \int_{X>0} e^{-tr[(T+R)X]} |\mathbf{X}|^{\alpha-\frac{1}{2}(p+1)} |I_p+\mathbf{X}|^{-\lambda} \, d\mathbf{X}$$

Since **T** is symmetric with arbitrary real elements and R is symmetric positive definite with constant elements, without loss of generality we may assume that **(T+R)** can be written as **FF′** where **F** is a pxp nonsigular matrix. But

$$tr[(\mathbf{T}+\mathbf{R})\mathbf{X}] = tr(\mathbf{FF'X}) = tr(\mathbf{F'XF})$$

Now using the transformation U = F′XF with Jacobian, $J(\mathbf{X}\to\mathbf{U}) = |\mathbf{F}|^{-(p+1)} = |\mathbf{FF'}|^{-\frac{1}{2}(p+1)}$, we get

$$L_x(T) = \frac{|\mathbf{R}|^{\alpha-\lambda}|\mathbf{T}+\mathbf{R}|^{\lambda-\alpha}}{\Gamma_p(\alpha,\lambda,\mathbf{R})} \int_{U>0} e^{-tr(U)} |\mathbf{U}|^{\alpha-\frac{1}{2}(p+1)} |(\mathbf{R}+\mathbf{T})+\mathbf{U}|^{-\lambda} \, d\mathbf{U}$$

$$= \frac{\Gamma_p(\alpha,\lambda,\mathbf{R}+\mathbf{T})}{\Gamma_p(\alpha,\lambda,\mathbf{R})} |I_p+\mathbf{TR}^{-1}|^{\lambda-\alpha} \qquad \textbf{Q.E.D}$$

Theorem (2)

Given that the pxp matrices X_1 and X_2 are independently distributed such that

$\mathbf{X}_1 \sim GG_p(\alpha_1,\lambda,\mathbf{R})$ Generalized matrix variate gamma distribution
$\mathbf{X}_{\backslash 2} \sim G_p(\alpha,\lambda,\mathbf{R})$ Ordinary matrix variate gamma distribution

where $\mathbf{X}_j = \mathbf{X}'_j > 0$, $Re(\alpha_j) > \frac{1}{2}(p-1)$, $j = 1,2$, and $\mathbf{R} = \mathbf{R}' > 0$ is a constant matrix. Then the matrix variate distribution of the random matrix

$$\mathbf{U} = \mathbf{X}_1 + \mathbf{X}_2 \qquad \ldots(4\text{-}3)$$

is given by

$$g(\mathbf{U}) = \frac{\Gamma_p(\alpha_1)\,|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1+\alpha_2)\,\Gamma_p(\alpha_1,\lambda,\mathbf{R})} e^{-tr(\mathbf{RU})}\,|\mathbf{U}|^{\alpha_1+\alpha_2-\frac{1}{2}(p+1)}$$

$${}_2F_1(\alpha_1,\lambda;\alpha_1+\alpha_2;-U) \quad U > 0\,,\ \mathrm{Re}(\alpha_1+\alpha_2) > \tfrac{1}{2}(p-1) \qquad \ldots(4\text{-}4)$$

Proof

Due to the independence of $\mathbf{X}_1$ and $\mathbf{X}_2$ the joint density of $\mathbf{X}_1$ and $\mathbf{X}_2$, denoted by $f(\mathbf{X}_1,\mathbf{X}_2)$, is the product of the marginal densities. That is

$$f(\mathbf{X}_1,\mathbf{X}_2) = f_1(\mathbf{X}_1)\, f_2(\mathbf{X}_2) =$$

$$\frac{|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1,\lambda,\mathbf{R})\,\Gamma_p(\alpha_2)} e^{-tr[\mathbf{R}(\mathbf{X}_1+\mathbf{X}_2)]}\,|\mathbf{X}_1|^{\alpha_1-\frac{1}{2}(p+1)}\,|\mathbf{I}_p+\mathbf{X}_1|^{-\lambda}|\mathbf{X}_2|^{\alpha_2-\frac{1}{2}(p+1)}$$

Put $\mathbf{U} = \mathbf{X}_1 + \mathbf{X}_2$ for fixed $\mathbf{X}_2$ which gives $\mathbf{X}_2 = \mathbf{U} - \mathbf{X}_1$, $J(\mathbf{X}_1 \to \mathbf{U}) = \mathbf{I}_p$ and $0 < \mathbf{X}_1 < \mathbf{U}$ (that is, $\mathbf{U} > 0$, $\mathbf{X}_1 > 0$, $\mathbf{U}-\mathbf{X}_1 > 0$). Then the density will become

$$f(X_1,U) = \frac{|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1,\lambda,\mathbf{R})\,\Gamma_p(\alpha_2)} e^{-tr[\mathbf{RU}]}$$

$$|\mathbf{X}_1|^{\alpha_1-\frac{1}{2}(p+1)}\,|\mathbf{I}_p+\mathbf{X}_1|^{-\lambda}|\mathbf{U}-\mathbf{X}_1|^{\alpha_2-\frac{1}{2}(p+1)}$$

for $U > 0$, $o < X_1 < U$. Note that the determinant

$$|\mathbf{U}-\mathbf{X}_1| = |\mathbf{U}|\,|\mathbf{I}_p - \mathbf{U}^{-1}\mathbf{X}_1| = |\mathbf{U}|\,|\mathbf{I}_p - \mathbf{X}_1\mathbf{U}^{-1}|$$
$$= |\mathbf{U}|\,|\mathbf{I}_p - \mathbf{U}^{-\frac{1}{2}}\mathbf{X}_1\,\mathbf{U}^{-\frac{1}{2}}|$$

where $U^{\frac{1}{2}}$ is the symmetric square root of $\mathbf{U} = \mathbf{U}' > 0$. Change $\mathbf{X}_1$ to $\mathbf{V} = \mathbf{U}^{-\frac{1}{2}}\,\mathbf{X}_1\,\mathbf{U}^{-\frac{1}{2}}$ for fixed $\mathbf{U}$. Then $\mathbf{X}_1 = \mathbf{U}^{\frac{1}{2}}\mathbf{V}\,\mathbf{U}^{\frac{1}{2}}$ and the Jacobian of the transformation, is $J(\mathbf{X}_1 \to \mathbf{V}) = |\mathbf{U}|^{\frac{1}{2}(p+1)}$. Then the joint density of $\mathbf{U}$ and $\mathbf{V}$ becomes

$$f(V,U) = \frac{|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1,\lambda,\mathbf{R})\,\Gamma_p(\alpha_2)} e^{-tr[\mathbf{RU}]}\,|\mathbf{U}|^{\alpha_1+\alpha_2-\frac{1}{2}(p+1)}$$

$$|\mathbf{V}|^{\alpha_1-\frac{1}{2}(p+1)}\,|\mathbf{I}_p-\mathbf{V}|^{\alpha_2-\frac{1}{2}(p+1)}\,|\mathbf{I}_p+\mathbf{U}\mathbf{V}|^{-\lambda}$$

for $0 < \mathbf{V} < \mathbf{I}$, $\mathbf{U} > 0$. Note that

$0 < \mathbf{X}_1 < \mathbf{U} \rightarrow 0 < \mathbf{U}^{-½}\mathbf{X}_1 \mathbf{U}^{-½} < \mathbf{I}, 0 < \mathbf{V} < \mathbf{I}.$

$$g(U) = \frac{|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1,\lambda,\mathbf{R})\,\Gamma_p(\alpha_2)} e^{-\text{tr}[\mathbf{RU})]} |\mathbf{U}|^{\alpha_1+\alpha_2-½(p+1)}$$

$$\int_{0<V<I_p} |\mathbf{V}|^{\alpha_1-½(p+1)} |\mathbf{I}_p-\mathbf{V}|^{\alpha_2-½(p+1)} |\mathbf{I}_p+\mathbf{UV}|^{-\lambda} \; \mathbf{dV}$$

$$= \frac{|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1,\lambda,\mathbf{R})\,\Gamma_p(\alpha_2)} e^{-\text{tr}[\mathbf{RU})]} |\mathbf{U}|^{\alpha_1+\alpha_2-½(p+1)}$$

$$\frac{\Gamma_p(\alpha_1)\,\Gamma_p(\alpha_2)}{\Gamma_p(\alpha_1,\alpha_2)} \; {}_2F_1(\alpha_1,\lambda;\, \alpha_1+\alpha_2;-\mathbf{U}) \qquad \ldots(4\text{-}5)$$

The above equality is obtained by using (2-13). Now, simplifying (4-5) we get the desired result (4-4).

Q.E.D

Remarks

(i) To prove that g(U) as givne in (4-4) is indeeed a p.d.f. note-from (2-24)-that

$$\int_{U>0} e^{-\text{tr}[\mathbf{RU})]} |\mathbf{U}|^{\alpha_1+\alpha_2-½(p+1)} \quad {}_2F_1(\alpha_1,\lambda;\, \alpha_1+\alpha_2;-\mathbf{U})\mathbf{dU}$$

$$= \Gamma_p(\alpha_1+\alpha_2)\, |\mathbf{R}|^{+\alpha_2} \qquad \Psi(\alpha_1;\alpha_1-\lambda+ ½(p+1);\mathbf{R}) \qquad \ldots(4\text{-}6)$$

Application of (4-6) and (3-6) shows that

$$\int_{U>0} \mathbf{g(U)dU} = \frac{\Gamma_p(\alpha_1)\,|\mathbf{R}|^{\alpha_1+\alpha_2-\lambda}}{\Gamma_p(\alpha_1+\alpha_2)\,\Gamma_p(\alpha_1)|\mathbf{R}|^{\alpha_1-\lambda}\,\Psi(\alpha_1;\alpha_1-\lambda+\frac{1}{2}(p+1);\mathbf{R})}$$

$$\Gamma_p(\alpha_1+\alpha_2)\, |\mathbf{R}|^{-\alpha_2} \qquad \Psi(\alpha_1;\alpha_1-\lambda+ ½(p+1);\mathbf{R})$$

$$= 1$$

as required.

(ii) In the special case when $\lambda = 0$ we can see-from (2-13)-that

$_2F_1(\alpha_1,0;\ \alpha_1+\alpha_2;-\mathbf{U})$

$$= \frac{\Gamma_p(\alpha_1)\,\Gamma_p(\alpha_2)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)} \int_{0<\mathbf{X}<\mathbf{I}_p} |\mathbf{X}|^{\alpha_1-\frac{1}{2}(p+1)}\ |\mathbf{I_p}-\mathbf{X}|^{\alpha_2-\frac{1}{2}(p+1)}\ \mathbf{dX}$$

$$= \frac{\Gamma_p(\alpha_1+\alpha_2)}{\Gamma_p(\alpha_1)\Gamma_p(\alpha_2)}\ \beta_p(\alpha_1,\alpha_2) = 1 \qquad \ldots(4\text{-}7)$$

Using (3-8) and (4-7) in (4-4) we find

$$g(\mathbf{U}) = \frac{|\mathbf{R}|^{\alpha_1+\alpha_2}}{\Gamma_p(\alpha_1+\alpha_2)}\ e^{-\mathrm{tr}[\mathbf{RU}]}\ |\mathbf{U}|^{\alpha_1+\alpha_2-\frac{1}{2}(p+1)} \qquad \ldots(4\text{-}8)$$

for $\mathbf{U} > 0$, $\mathrm{Re}(\alpha_1+\alpha_2) > \frac{1}{2}(p-1)$. Therefore, if the pxp matrices $\mathbf{X}_j$, $j = 1,2$ are independetly distributed such that

$\mathbf{X}_j \sim \mathbf{G}_p(\alpha_j,\mathbf{R})$ ordinary matrix variate gamma distribution

then

$$\mathbf{U} = \mathbf{X}_1 + \mathbf{X}_2 \sim \mathbf{G}_p(\alpha_1+\alpha_2,\mathbf{R})$$

Theorem (3)

Let $\mathbf{X}_1 \sim \mathbf{GG}_p(\alpha_1,\lambda,\mathbf{I_p})$ and $\mathbf{X}_2 \sim \mathbf{G}_p(\alpha_2,\mathbf{I_p})$ be independent. Define

$$\mathbf{V} = \mathbf{X}_1^{-\frac{1}{2}}\ \mathbf{X}_2\ \mathbf{X}_1^{-\frac{1}{2}} \qquad \ldots(4\text{-}9)$$

Where $\mathbf{X}_1^{\frac{1}{2}}$ is a symmetric square root of $\mathbf{X}_1$. Then, the p.d.f. of $\mathbf{V}$ is given by

$$\frac{\Gamma_p(\alpha_1+\alpha_2)}{\Gamma_p(\alpha_1,\lambda,\mathbf{I}_p)\Gamma_p(\alpha_2)}|\mathbf{V}|^{\alpha_2-\frac{1}{2}(p+1)}\ \Psi(\alpha_1+\alpha_2;\alpha_1+\alpha_2-\lambda+\tfrac{1}{2}(p+1);\mathbf{I_p}+\mathbf{V})$$

$$\mathbf{V} > 0,\ \ \mathrm{Re}(\alpha_1,\alpha_2)\ \tfrac{1}{2}(p-1) \qquad \ldots(4\text{-}10)$$

Proof:

The joint p.d.f of $\mathbf{X}_1$ and $\mathbf{X}_2$ is

$$f(\mathbf{X}_1,\mathbf{X}_2) = \frac{e^{-\mathrm{tr}\,(\mathbf{X}_1+\mathbf{X}_2)}}{\Gamma_p(\alpha_1,\lambda,\mathbf{I}_p)\,\Gamma_p(\alpha_2)} \quad |\mathbf{X}_1|^{\alpha_1-\frac{1}{2}(p+1)}\,|\mathbf{I}_p+\mathbf{X}_1|^{-\lambda}|\mathbf{X}_2|^{\alpha_2-\frac{1}{2}(p+1)}$$

$$\mathbf{X}_1 > 0,\ \mathbf{X}_2 > 0$$

Transforming $\mathbf{V} = \mathbf{X}_1^{-\frac{1}{2}}\,\mathbf{X}_2\,\mathbf{X}_1^{-\frac{1}{2}}$, with Jacobian $J(\mathbf{X}_1,\mathbf{X}_2 \rightarrow \mathbf{V},\mathbf{X}_1) = |\mathbf{X}_1|^{\frac{1}{2}(p+1)}$, we get the joint p.d.f. of $\mathbf{X}_1$ and $\mathbf{V}$ as

$$f(\mathbf{X}_1,\mathbf{V}) = \frac{1}{\Gamma_p(\alpha_1,\lambda,\mathbf{I}_p)\Gamma_p(\alpha_2)}\ |\mathbf{V}|^{\alpha_2-\frac{1}{2}(p+1)}$$

$$e^{-\mathrm{tr}[(\mathbf{I}p+\mathbf{V})\mathbf{X}_1]}\ |\mathbf{X}_1|^{\alpha_2+\alpha_2-\frac{1}{2}(p+1)}\ |\mathbf{I}_p+\mathbf{X}_1|^{-\lambda} \quad \mathbf{X}_1 > 0,\ \mathbf{V} > 0 \qquad \ldots(4\text{-}11)$$

Now, integrating out $\mathbf{X}_1$ from (4-11), using (2-18), we get

$$g(\mathbf{V}) = \frac{|\mathbf{V}|^{\alpha_2-1/2(p+1)}}{\Gamma_p(\alpha_1,\lambda,\mathbf{I}_p)\,\Gamma_p(\alpha_2)} \int_{\mathbf{X}_1>0} e^{-\mathrm{tr}[(\mathbf{I}_p+\mathbf{V})\mathbf{X}_1]}\,|\mathbf{X}_1|^{\alpha_1+\alpha_2-\frac{1}{2}(p+1)}\ |\mathbf{I}_p+\mathbf{X}_1|^{-\lambda}\ d\mathbf{X}_1$$

$$= \frac{|\mathbf{V}|^{\alpha_2-1/2(p+1)}}{\Gamma_p(\alpha_1,\lambda,\mathbf{I}_p)\,\Gamma_p(\alpha_2)}\,\Gamma_p(\alpha_1+\alpha_2)\quad \Psi(\alpha_1+\alpha_2;\ \alpha_1+\alpha_2-\lambda+\tfrac{1}{2}(p+1);\mathbf{I}_p+\mathbf{V})$$

which is the required result stated in (4-10).

Q.E.D.

Theorem (4)

Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be independently distributed as $GG_p(n_1,\lambda,\mathbf{I}_p)$ and $G_p(n_2,.\mathbf{I}_p)$ respectivley. Let

$$\mathbf{F} = (\mathbf{W}_1+\mathbf{W}_2)^{-\frac{1}{2}}\quad \mathbf{W}_1(\mathbf{W}_1+\mathbf{W}_2)^{-\frac{1}{2}} \qquad \ldots(4\text{-}12)$$

Where $(\mathbf{W}_1+\mathbf{W}_2)^{\frac{1}{2}}$ is any nonsingular factorization of $\mathbf{W}_1+\mathbf{W}_2$ in the sense $\mathbf{W}_1+\mathbf{W}_2 = (\mathbf{W}_1+\mathbf{W}_2)^{\frac{1}{2}}[(\mathbf{W}_1+\mathbf{W}_2)^{\frac{1}{2}}]'$. Then the p.d.f. of $\mathbf{F}$ is given

$$\frac{\Gamma_p(n_1+n_2)}{\Gamma_p(n_1,\lambda,I_p)\Gamma_p(n_2)}\ |\mathbf{F}|^{-n_2-\frac{1}{2}(p+1)}\ |\mathbf{I}_p-\mathbf{F}|^{n_2-\frac{1}{2}(p+1)}$$

$$\Psi(n_1+n_2;\ n_1+n_1-\lambda+\tfrac{1}{2}(p+1);\ \mathbf{F}^{-1}) \qquad 0 < \mathbf{F} < \mathbf{I}_p$$

Proof:

The joint p.d.f. of $\mathbf{W}_1$ and $\mathbf{W}_2$ is given by

$$f(\mathbf{W}_1,\mathbf{W}_2) = \frac{1}{\Gamma_p(n_1,\lambda,I_p)\Gamma_p(n_2)} e^{-tr(\mathbf{W}_1+\mathbf{W}_2)}$$

$$|\mathbf{W}_1|^{n_1-\frac{1}{2}(p+1)} |\mathbf{I}+\mathbf{W}_1|^{-\lambda} |\mathbf{W}_2|^{n_2-\frac{1}{2}(p+1)} \quad \mathbf{W}_1 > 0,\ \mathbf{W}_2 > 0$$

Let $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$ and $\mathbf{W}_1 = \mathbf{W}_1$. Then the joint p.d.f. of $\mathbf{W}$ and $\mathbf{W}_1$ is given by

$$f(\mathbf{W},\mathbf{W}_1) = \frac{e^{-tr(\mathbf{W})}}{\Gamma_p(n_1,\lambda,I_p)\Gamma_p(n_2)} |\mathbf{W}_1|^{n_1-\frac{1}{2}(p+1)} |\mathbf{I}+\mathbf{W}_1|^{-\lambda}$$

$$.|\mathbf{W}-\mathbf{W}_1|^{n_2-\frac{1}{2}(p+1)} \qquad \mathbf{W} > 0,\ 0 < \mathbf{W}_1 < \mathbf{W}$$

Let $\mathbf{F} = \mathbf{W}^{-\frac{1}{2}}\ \mathbf{W}_1\ \mathbf{W}^{-\frac{1}{2}}$ where $\mathbf{W} = \mathbf{W}^{\frac{1}{2}}\ \mathbf{W}^{\frac{1}{2}\prime}$. Then the joint p.d.f. of $\mathbf{F}$ and $\mathbf{W}$ [since $J(\mathbf{W}_1 \to \mathbf{F}) = |\mathbf{W}|^{\frac{1}{2}(p+1)}$] is given by

$$g(\mathbf{F},\mathbf{W}) = \frac{1}{\Gamma_p(n_1,\lambda,I_p)\Gamma_p(n_2)} |\mathbf{F}|^{n_1-\frac{1}{2}(p+1)} |\mathbf{I_p} - \mathbf{F}|^{n_2-\frac{1}{2}(p+1)}$$

$$e^{-tr(\mathbf{W})} |\mathbf{W}|^{n_1+n_2-\frac{1}{2}(p+1)} |\mathbf{I}_p + \mathbf{W}^{\frac{1}{2}}\mathbf{F}\ \mathbf{W}^{\frac{1}{2}}|^{-\lambda}$$

$$0 < \mathbf{F} < \mathbf{I}_p,\ \mathbf{W} > 0, \qquad \ldots(4\text{-}14)$$

Integrating out $\mathbf{W}$ from (4-14), we get

$$g(F) = \frac{1}{\Gamma_p(n_1,\lambda,I_p)\Gamma_p(n_2)} |\mathbf{F}|^{n_1-\frac{1}{2}(p+1)-\lambda} |\mathbf{I_p} - \mathbf{F}|^{n_2-\frac{1}{2}(p+1)}$$

$$\int_{\mathbf{W}>0} e^{-tr(\mathbf{W})} |\mathbf{W}_1|^{n_1+n_2-\frac{1}{2}(p+1)} |\mathbf{F}^{-1}+\mathbf{W}|^{-\lambda}\ d\mathbf{W}$$

$$= \frac{1}{\Gamma_p(n_1,\lambda,I_p)\Gamma_p(n_2)} |\mathbf{F}|^{n_1-\frac{1}{2}(p+1)-\lambda} |\mathbf{I_p} - \mathbf{F}|^{n_2-\frac{1}{2}(p+1)}$$

$$.\ \Gamma_p(n_1+n_2)\ |\mathbf{F}^{-1}|^{n_1+n_2-\lambda}\ \Psi(n_1+n_2; n_1+n_2-\lambda+\tfrac{1}{2}(p+1); \mathbf{F}^{-1})$$

where the last integral has been evaluated by using the second equality of (2-18). This completes the proof.

Q.E.D.

Remarks

In the special case when $\lambda = 0$ the results of Theorems (3) and (4) can be simplified as follows:

(i) In the p.d.f. (4-10) of Theorem (3)-with $\lambda = 0$ – we have

$$\Gamma_p(\alpha_1, 0, I_p) = \Gamma_p(\alpha_1)$$

$$\Psi(\alpha_1+\alpha_2; \alpha_1+\alpha_2+½(p+1); \mathbf{I}_p+\mathbf{V})$$

$$= \frac{1}{\Gamma_p(\alpha_1,\alpha_2)} \int_{S>0} e^{-\underline{tr}[(\mathbf{I}_p+\mathbf{V})\mathbf{S}]} |\mathbf{S}|^{\alpha_1+\alpha_2-½(p+1)} \, d\mathbf{S}$$

$$= \frac{1}{\Gamma_p(\alpha_1,\alpha_2)} (\mathbf{I}_p + \mathbf{V})^{-(\alpha_1+\alpha_2)} \Gamma_p(\alpha_1+\alpha_2)$$

$$= (\mathbf{I}_p + \mathbf{V})^{-(\alpha_1+\alpha_2)}$$

Thus the p.d.f. (4-10) of the random matrix $\mathbf{V}$ (4-9)- when $\lambda = 0$ – takes the form

$$\frac{\Gamma_p(\alpha_1,\alpha_2)}{\Gamma_p(\alpha_1)\Gamma(\alpha_2)} |\mathbf{V}|^{\alpha_2-½(p+1)} |\mathbf{I}_p + \mathbf{V}|^{-(\alpha_1+\alpha_2)} \qquad \mathbf{V} > 0$$

...(4-15)

which is the p.d.f. of the matrix variate Beta Type II distribution.

(ii) In the p.d.f. (4-13) of Theorem (4)-with $\lambda = 0$ - we have

$$\Gamma_p(n_1, 0, I_p) = \Gamma_p(n_1)$$

$$\Psi(n_1+n_2; n_1+n_2+½(p+1); \mathbf{F}^{-1})$$

$$= \frac{1}{\Gamma_p(\mathbf{n}_1,\mathbf{n}_2)} \int_{V>0} e^{-\underline{tr}[(\mathbf{F}^{-1}\,\mathbf{S}]} |\mathbf{S}|^{n_1+n_2-½(p+1)} \, d\mathbf{S}$$

$$= \frac{1}{\Gamma_p(n_1,n_2)} \; |F^{-1}|^{-(n_1+n_2)} \; \Gamma_p(n_1+n_2)$$

$$= |F|^{n_1+n_2}$$

Thus the p.d.f. (4-13) of the random matrix **F**(4-12)- when $\lambda = 0$ – takes the form

$$= \frac{\Gamma_p(n_1,n_2)}{\Gamma_p(n_1)\Gamma(n_2)} \; |F|^{n_1-½(p+1)} \; |I_p - F|^{n_2-½(p+1)}$$

$$0 < F < I_p$$

which is the p.d.f. of the matrix variate Beta Type II distribution.

§ 5. SOME PROBABILITY DISTRIBUTIONS CONNECTED WITH GENERALIZED MATRIX VARIATE GAMMA DISTRIBUTION

Definition (8):

A random matrix **V** (pxp) is said to follow a generalized matrix variate inverted gamma distribution, denoted as $V \sim GIG_p(m,\lambda,R)$, if its p.d.f. is

$$\frac{|R|^{m-1/2(p+1)-\lambda}}{\Gamma_p(m-1/2(p+1),\lambda,R)} e^{-tr(RV^{-1})} \; |V|^{\lambda-m} \; |I_p+V|^{-\lambda}$$

$$V > 0,\ R(pxp) > 0,\ m > p \qquad \ldots(5\text{-}1)$$

The relation between the generalized matrix variate Gamma and Generalized matrix variate inverted Gamma distributions is given in the following theorem.

Theorem (5)

Let $V \sim GIG_p(m,\lambda,R)$, then $V^{-1} \sim GG_p(m-½(p+1),\lambda,R)$

Proof

The density of **V** is as given in (5-1). Transforming $S = V^{-1}$ with Jacobian $J(V \rightarrow S) = |S|^{-(p+1)}$, we get the density of **S** as

$$f(\mathbf{S}) = \frac{|\mathbf{R}|^{m-1/2(p+1)-\lambda}}{\Gamma_p[m-1/2(p+1),\lambda,\mathbf{R}]} e^{-\underline{tr}(\mathbf{RS})} |\mathbf{S}^{-1}|^{\lambda-m} |\mathbf{I}_p+\mathbf{S}^{-1}|^{\lambda} |\mathbf{S}|^{-(p+1)}$$

$$= \frac{|\mathbf{R}|^{m-1/2(p+1)-\lambda}}{\Gamma_p[m-1/2(p+1),\lambda,\mathbf{R}]} e^{-\underline{tr}(\mathbf{RS})} |\mathbf{S}|^{m-\lambda} |\mathbf{I}_p+\mathbf{S}|^{-\lambda} |\mathbf{S}|^{\lambda} |\mathbf{S}|^{-(p+1)}$$

$$= \frac{|\mathbf{R}|^{m-1/2(p+1)-\lambda}}{\Gamma_p[m-1/2(p+1),\lambda,\mathbf{R}]} e^{-\underline{tr}(\mathbf{RS})} |\mathbf{S}|^{[m-½(p+1)]-½(p+1)} |\mathbf{I}_p+\mathbf{S}|^{-\lambda}$$

which is the p.d.f. of $GG_p[m-½(p+1),\lambda,\mathbf{R}]$

Q.E.D.

Theorem (6)

Let $\mathbf{V} \sim GIG_p(m,\lambda,\mathbf{I_p})$, $\mathbf{W} \sim G_p(n,\mathbf{I_p})$ be independent. Define

$$\mathbf{Z} = \mathbf{V}^{½}\,\mathbf{W}\,\mathbf{V}^{½} \qquad \text{...(5-2)}$$

Where $\mathbf{V}^{½}$ is a symmetric square root of V. Then, the p.d.f. of Z is given by

$$\frac{\Gamma_p[(m+n)-1/2(p+1),\lambda,\mathbf{I}_p]}{\Gamma_p(n)\,\Gamma_p[m-1/2(p+1),\lambda,\mathbf{I}_p]} \quad \frac{|\mathbf{Z}|^{n-1/2(p+1)}}{|\mathbf{I}_p+\mathbf{Z}|^{(m+n)-1/2(p+1)-\lambda}}$$

$$\mathbf{Z} > 0 \qquad \text{...(5-3)}$$

or equivalently by

$$\frac{\Gamma_p[m+n-1/2(p+1)]}{\Gamma_p(n)\,\Gamma_p[m-1/2(p+1),\lambda,\mathbf{I}_p]} |\mathbf{Z}|^{n-½(p+1)}\, \Psi(m+n-½(p+1);m+n-\lambda;\mathbf{I}_p+\mathbf{Z})$$

$$\mathbf{Z} > 0 \qquad \text{...(5-4)}$$

Proof

The joint p.d.f. of **V** and **W** is given by

$$f(\mathbf{V},\mathbf{W}) = \frac{1}{\Gamma_p(n)\,\Gamma_p[m-1/2(p+1),\lambda,\mathbf{I}_p]} e^{-\underline{tr}(\mathbf{W}+\mathbf{V}^{-1})}$$

$$|\mathbf{W}|^{n-½(p+1)} |\mathbf{V}|^{\lambda-m} |\mathbf{I}_p+\mathbf{V}|^{-\lambda}$$

for $m > p$, $n > ½(p-1)$, $\mathbf{W} > 0$, and $\mathbf{V} > 0$.

Transforming $\mathbf{Z} = \mathbf{V}^{½}\, \mathbf{W}\, \mathbf{V}^{½}$, with Jacobian $J(\mathbf{W}\rightarrow\mathbf{V}) = |\mathbf{V}|^{-½(p+1)}$, we get the joint p.d.f. of $\mathbf{V}$ and $\mathbf{Z}$ as

$$g(\mathbf{V},\mathbf{Z}) = \frac{1}{\Gamma_p(n)\,\Gamma_p[m-1/2(p+1),\lambda,\mathbf{I}_p]} |\mathbf{Z}|^{n-½(p+1)}$$

$$e^{-tr[(\mathbf{I}_p+\mathbf{Z})\mathbf{V}^{-1}]}\, |\mathbf{V}|^{\lambda-(m+n)}\, |\mathbf{I}_p+\mathbf{V}|^{-\lambda}$$

$$\mathbf{V} > 0, \text{ and } \mathbf{Z} > 0 \qquad \text{...(5-5)}$$

Now to obtain the marginal p.d.f. of Z, we need to evaluate

$$\Delta = \int_{\mathbf{V}>0} e^{-tr[(\mathbf{I}_p+\mathbf{Z})\mathbf{V}^{-1}]}\, |\mathbf{V}|^{\lambda-(m+n)}\, |\mathbf{I}_p+\mathbf{V}|^{-\lambda}\, d\mathbf{V} \qquad \text{...(5-6)}$$

Substituting in (5-6), $\mathbf{S} = \mathbf{V}^{-1}$ with the Jacobian $J(\mathbf{V}\rightarrow\mathbf{S}) = |\mathbf{S}|^{-(p+1)}$, we get

$$\Delta = \int_{\mathbf{V}>0} e^{-tr[(\mathbf{I}_p+\mathbf{Z})\mathbf{S}]}\, |\mathbf{S}|^{[(m+n)-½(p+1)]-½(p+1)}\, |\mathbf{I}_p+\mathbf{S}|^{-\lambda}\, d\mathbf{S}$$

$$= \Gamma_p[(m+n)-½(p+1),\lambda,\mathbf{I}_p+\mathbf{Z}] \,/\, |\mathbf{I}_p+\mathbf{Z}|^{(m+n)-½(p+1)-\lambda} \qquad \text{...(5-7)}$$

$$= \Gamma_p[(m+n)-½(p+1)]\quad \Psi(m+n-½(p+1);m+n-\lambda;\mathbf{I}_p+\mathbf{Z}) \qquad \text{...(5-8)}$$

wherein (3-5) and (3-6) have been used. Using (5-5) together with (5-7) and (5-8) we get the required results (5-3) and (5-4).

Q.E.D.

In the sequel, we propose a distribution connected with the generalized matrix variate gamma distribution. This distribution may be called "The Generalized Matrix Variate-t Distribution".

Definition (9)

The random matrix $\mathbf{T}(p\text{x}m)$ is said to have a Generalized Matrix Varite t-Distribution, denoted as $\mathbf{T} \sim \mathbf{T}_{p,m}(n,\mathbf{M},\Sigma,\Omega,\Psi)$, if its p.d.f. is given be

$$\frac{\Gamma_p[1/2(n+m+p-1,\lambda,\Psi]}{(2\pi)^{1/2m\,p}\Gamma_p[1/2(n+p-1),\lambda,\sum\ ]}\ |\Sigma|^{-½m}$$

$$|\Omega|^{-½p}\ |I_p+½\ \Sigma^{-1}(\mathbf{T}-\mathbf{M})\ \Omega^{-1}(\mathbf{T}-\mathbf{M})'\ |^{\lambda-½(n+m+p-1)} \qquad \ldots(5\text{-}9)$$

where

$$\Psi = \Sigma + ½\ (\mathbf{T}-\mathbf{M})\ \Omega^{-1}(\mathbf{T}-\mathbf{M})'$$

and

$$T \in R^{pxm}\ ,\ M\ \in R^{pxm},\ \ \Omega(mxm) > 0,\ \Sigma(pxp) > 0 \text{ and } n > 0.$$

This distribution can be derived in a simple manner as shown in the following theorem♣.

Theorem (7)

Let $V \sim GG_p(½(n+p-1),\lambda,\Sigma)$, independent of $\mathbf{X} \sim N_{p,m}(\mathbf{O},\mathbf{I}_p \otimes \Omega)$. Define

$$\mathbf{T} = (\mathbf{V}^{-½})'\mathbf{X}+\mathbf{M} \qquad \ldots(5\text{-}11)$$

Where M(pxm) is a constant matrix, and $V^{½}\,(V^{½}) = V$**. Then,**
$T \sim T_{p,m}(\mathbf{n},M,\Sigma,\Omega,\Psi)$**.**

Proof

The joint p.d.f. of V **and** X **is given by**

$$f(\mathbf{V},\mathbf{X}) = \frac{(2\pi)^{-1/2m\,p}\left|\sum\ \right|^{1/2(n+p-1)-\lambda}\left|\Omega\right|^{-1/2\,p}}{\Gamma_p[1/2(n+p-1),\lambda,\sum\ ]}$$

$$e^{-tr[\Sigma V+½X\ \Omega^{-1}X']}\ |\mathbf{V}|^{½(n-2)}\ \ |I_p+V|^{-\lambda} \quad V > 0, \text{ and } \mathbf{X} \in R^{pxm}$$

♣ The random matrix X(pxm) is said to have a matrix variate normal distribution with mean matrix M (pxm) and covariance matrix $\Sigma \otimes \Omega$ if its p.d.f. is given by $(2\pi)^{-½mp}\ |\Sigma|^{-½m}\ \ |\Omega|^{-½p}\ \ e^{-½tr[\Sigma^{-1}(X-M)\Omega^{-1}(X-M)']}$ where $\mathbf{X} \in \mathbf{R}^{pxm}$, $\mathbf{M} \in \mathbf{R}^{pxm}$, $\Sigma(pxp) > 0$, and $\Omega(mxm) > 0$. We shall denote this by $\mathbf{X} \sim \mathbf{N}_{p,m}(\mathbf{M},\Sigma \otimes \Omega)$.

Now, let $\mathbf{T} = (\mathbf{V}^{-1/2})'\mathbf{X} + \mathbf{M}$. The Jacobian of the transformation is $J(\mathbf{X}\rightarrow\mathbf{T}) = |\mathbf{V}|^{1/2 m}$. Subsitituting for **X** in terms of **T** in the joint p.d.f. of **V** and **X**, and multiplying the resulting expression by $J(\mathbf{X}\rightarrow\mathbf{T})$, we get the joint p.d.f. of **T** and **V** as

$$g(\mathbf{T},\mathbf{V}) = \frac{(2\pi)^{-1/2 m p} \left|\sum\right|^{1/2(n+p-1)-\lambda} |\Omega|^{-1/2p}}{\Gamma_p[1/2(n+p-1),\lambda,\sum]} |\mathbf{V}|^{1/2(n+m-2)} |\mathbf{I}_p+\mathbf{V}|^{-\lambda}$$

$$e^{-tr}\{[\Sigma+1/2(T-M)\,\Omega^{-1}(\mathbf{T}-\mathbf{M})']V\}$$

Now, integrating out **V** using the generalized matrix variate gamma integral (3-5) the p.d.f. of **T** is obtained as

$$h(\mathbf{T}) = \frac{\Gamma_p[1/2(n+m+p-1),\lambda,\Psi]}{(2\pi)^{1/2 m p}\,\Gamma_p[1/2(n+p-1),\lambda,\sum]} |\Sigma|^{-1/2 m} |\Omega|^{-1/2 p}$$

$$|\mathbf{I}_p+1/2\,\Sigma^{-1}(T-M)\,\Omega^{-1}(\mathbf{T}-\mathbf{M})'|^{\lambda-1/2(n+m+p-1)}$$

where Ψ is as defined in (5-10). This completes the proof.

Q.E.D

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