@ISSR, Cairo University, 2008

On Double Stochastic Lumped Markov Chains

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Abstract

In this paper, the problem of aggregation Markov chains has been considered. The necessary and sufficient conditions of the transition probability matrix of the original Markov chain to be a doubly stochastic matrix when the transition probability matrix of the lumped chain was doubly stochastic have been given for the lumping Markov Chain for all (certain) partitions within isomorphic with respect to the cases (k, k+1), (k, k+2) and (2, k). Also, special cases have been introduced for each case, for each partition.

Key Words

Markov Chains – Aggregation – Lumped Chain – Doubly Stochastic Matrix.

Introduction

In (1958), C.J.Burke and M.Rosenblatt gave a necessary and
cient condition for the lumped chain $Y(t)$
original Markov Chain $X(t)$ is a reversibl
is a continuous time Markov process.
essary and sufficient condition for $Y(t)$
has a continuous time parameter. So, Bo
ieffer's condition. He gave necessary an

Where,

1- $\sum_{i=1}^{2}\lambda_{ij}=1,$

2- $\sum_{i=3}^{4}\lambda_{ij}=1,$

an exponential type process to be Markov. A.Moneim and F.Leysieffer (1984) gave conditions for $Y(t)$ to be a Markov Chain when $X(t)$ is not irreducible. These conditions are given in terms of an interrelationship between two partitions of the state space of the Markov Chain $X(t)$. R.V.Erickson (1970), determined necessary and sufficient conditions under which an arbitrary finite state process is a combining state result of a Markov Chain. A.B.Volidis (1985) gave the sufficient conditions for $\ell(P,A)=P^*$, where P is the transition probability matrix (t.p.m) of an absorbing Markov Chain $X(t)$ and P^* is the t.p.m of the lumped process $Y(t)$ which is also an absorbing chain. He expressed the fundamental matrix of $Y(t)$ in terms of the fundamental matrix of the original chain $X(t)$. Peng (1996) obtained the necessary and sufficient condition for weak lumpability and the characterization of the set of initial starting vectors which make it lumpable, under a mild condition Γ on the transition probability matrix. Also, a similar result is obtained for those transition probability matrices without the restriction of condition Γ.

Ping (2004) presented an analysis of the convergence behavior of Markovian iteration matrices that is for an irreducible and homogeneous Markov Chain a necessary and sufficient condition for convergence in one iteration is that the iteration matrix have rank one. Also, he investigated the relationship among lumpability, weak lumpability, quasi-lumpability and near complete decomposability. Jacobi and Gornerup (2007) presented the necessary and sufficient conditions for identifying strong lumpability in Markov Chains. They showed that the states in a lump necessarily correspond to identical elements in eigenvectors of the dual transition matrix.

Assume that $Y(t)$ is the result of an a

ain $X(t)$ of higher dimension under given

l unknown t.p.m P with t.p.m P^* which is

$\sum_k p_{ik}=\sum_k p_{kj}=1,\ \ p_{ij}\ge 0,\ \ \forall i,j$). So, in the

oduce the necessary and sufficient condi

References

1- Abdel Moneim,A.M. and

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2- Boudreau,P.E. (1968),"

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3- Burk C. J. and Rosenb

t.p.m P where $\ell(P,A)=P^*$ to be a doubly stochastic matrix for all and certain partitions within isomorphic.

2. The General Case m=k, n=k+1

In this case, we have just only one partition within isomorphic take the form:

$$A=\{\{1\},\{2\},\ldots,\{k,k+1\}\}$$

The t.p.m of the lumped chain $Y(t)$ will be,

$$P^*=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \downarrow & \downarrow & \cdots & \downarrow \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \qquad \textbf{(2.1)}$$

Where, $\sum_k a_{ik}=\sum_k a_{kj}=1,\quad a_{ij}\geq 0,\quad \forall i,j$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k+1)\times(k+1)$, where $\ell(P,A)=P^*$ will be,

$$P=\begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1k}a_{1k} & \lambda_{1(k+1)}a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2k}a_{2k} & \lambda_{2(k+1)}a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & \lambda_{kk}a_{kk} & \lambda_{k(k+1)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)k}a_{kk} & \lambda_{(k+1)(k+1)}a_{kk} \end{bmatrix}, \qquad \textbf{(2.2)}$$

Where, $\sum_{j=k}^{k+1}\lambda_{ij}=1,\quad \forall i=1,2,\ldots,k$

Since, the t.p.m P^* is a double stochastic matrix, then after very simple steps the necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix will be:

$$\left.\begin{array}{l} \cdot\sum_{i=1}^{k-1}a_{ij}=1,\qquad j=1,2,\ldots,k-1 \\ \cdot\sum_{j=k}^{k+1}a_{ij}=0,\quad \forall i=1,2,\ldots,k-1 \Leftrightarrow a_{ij}=0,\quad \forall i=1,2,\ldots,k-1\ \ \&\ \ \forall j=k,k+1 \\ \cdot\sum_{i=k}^{k+1}a_{ij}=0,\quad \forall j=1,2,\ldots,k-1 \Leftrightarrow a_{ij}=0,\quad \forall j=1,2,\ldots,k-1\ \ \&\ \ \forall i=k,k+1 \\ \cdot\sum_r\lambda_{ir}=\sum_r\lambda_{rj}=1,\quad \forall i,j=k,k+1 \end{array}\right\} \textbf{(2.3)}$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} R & O \\ O' & T \end{bmatrix} \tag{2.4}$$

Where,

$$\left.\begin{aligned}
&1)\ R = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} \\ a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)(k-1)} \end{bmatrix}_{(k-1)(k-1)} \\
&2)\ O \text{ is the zero matrix of order } (k-1)\times 2 \\
&3)\ T = \begin{bmatrix} \lambda_{kk} & \lambda_{k(k+1)} \\ \lambda_{(k+1)k} & \lambda_{(k+1)(k+1)} \end{bmatrix}_{(2\times 2)}
\end{aligned}\right\} \tag{2.5}$$

2.1 The Special Case m=4, n=5

The form of the t.p.m P^* such that $\ell(P,A) = P^*$ is:

$$P^* = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Where, $\sum_{k=1}^{4} a_{ik} = \sum_{k=1}^{4} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i,j$

So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{44} & \lambda_{45} \\ 0 & 0 & 0 & \lambda_{54} & \lambda_{55} \end{bmatrix}$$

3. The General Case m=k, n=k+2

In this case, we have only two partitions within isomorphic:

$$A_1 = \{\{1\},\{2\},\ldots,\{k,k+1,k+2\}\}$$

$$A_2 = \{\{1\},\{2\},\ldots,\{k-1,k\},\{k+1,k+2\}\}$$

Where, $k \geq 3$.

The t.p.m P^* of the lumped chain $Y(t)$ will take the same form as in equation (2.1).

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k+2)\times(k+2)$, where $\ell(P,A_1)=P^*$ will be,

$$P=\begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1k}a_{1k} & \lambda_{1(k+1)}a_{1k} & \lambda_{1(k+2)}a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2k}a_{2k} & \lambda_{2(k+1)}a_{2k} & \lambda_{2(k+2)}a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & \lambda_{kk}a_{kk} & \lambda_{k(k+1)}a_{kk} & \lambda_{k(k+2)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)k}a_{kk} & \lambda_{(k+1)(k+1)}a_{kk} & \lambda_{(k+1)(k+2)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+2)k}a_{kk} & \lambda_{(k+2)(k+1)}a_{kk} & \lambda_{(k+2)(k+2)}a_{kk} \end{bmatrix}, \quad (3.1)$$

Where, $\sum_{j=k}^{k+2}\lambda_{ij}=1, \quad \forall i=1,2,\ldots,k$

a- ***The partition*** $A_1=\{\{1\},\{2\},\ldots,\{k,k+1,k+2\}\}$

The necessary and sufficient conditions for the t.p.m P to be A double stochastic matrix with respect to the partition A_1 within isomorphic will be:

$$\left.\begin{array}{l} 1\text{-} \sum_{i=1}^{k-1}a_{ij}=1, \qquad j=1,2,\ldots,k-1 \\ 2\text{-} \sum_{j=k}^{k+2}a_{ij}=0, \quad \forall i=1,2,\ldots,k-1, \Leftrightarrow \\ \qquad a_{ij}=0, \quad \forall i=1,2,\ldots,k-1 \ \ \& \ \ \forall j=k,k+1,k+2 \\ 3\text{-} \sum_{i=k}^{k+2}a_{ij}=0, \quad \forall j=1,2,\ldots,k-1, \Leftrightarrow \\ \qquad a_{ij}=0, \quad \forall j=1,2,\ldots,k-1 \ \ \& \ \ \forall i=k,k+1,k+2 \\ 4\text{-} \sum_r \lambda_{ir}=\sum_r \lambda_{rj}=1, \quad \forall i,j=k,k+1,k+2 \end{array}\right\} \quad (3.2)$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P=\begin{bmatrix} R & O \\ O' & T \end{bmatrix} \quad (3.3)$$

Where,

1) R is the same matrix as in (2.5).
2) O is the zero matrix of order $(k-1)\times 3$
3) $T=\begin{bmatrix} \lambda_{kk} & \lambda_{k(k+1)} & \lambda_{k(k+2)} \\ \lambda_{(k+1)k} & \lambda_{(k+1)(k+1)} & \lambda_{(k+1)(k+2)} \\ \lambda_{(k+2)k} & \lambda_{(k+2)(k+1)} & \lambda_{(k+2)(k+2)} \end{bmatrix}_{(3\times 3)}$

(3.4)

3.1 The Special Case m=3, n=5

The form of the t.p.m P^* such that $\ell(P, A_1) = P^*$ is:

$$P^* = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \textbf{(3.5)}$$

Where, $\sum_{k=1}^{3} a_{ik} = \sum_{k=1}^{3} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & \lambda_{43} & \lambda_{44} & \lambda_{45} \\ 0 & 0 & \lambda_{53} & \lambda_{54} & \lambda_{55} \end{bmatrix}$$

b- The partition $A_2 = \{\{1\},\{2\},\ldots,\{k-1,k\},\{k+1,k+2\}\}$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k+2)\times(k+2)$, where $\ell(P, A_2) = P^*$ will be,

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1(k-1)} a_{1(k-1)} & \lambda_{1k} a_{1(k-1)} & \lambda_{1(k+1)} a_{1k} & \lambda_{1(k+2)} a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2(k-1)} a_{2(k-1)} & \lambda_{2k} a_{2(k-1)} & \lambda_{2(k+1)} a_{2k} & \lambda_{2(k+2)} a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & \lambda_{(k-1)(k-1)} a_{(k-1)(k-1)} & \lambda_{(k-1)k} a_{(k-1)(k-1)} & \lambda_{(k-1)(k+1)} a_{(k-1)k} & \lambda_{(k-1)(k+2)} a_{(k-1)k} \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & \lambda_{k(k-1)} a_{(k-1)(k-1)} & \lambda_{kk} a_{(k-1)(k-1)} & \lambda_{k(k+1)} a_{(k-1)k} & \lambda_{k(k+2)} a_{(k-1)k} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)(k-1)} a_{k(k-1)} & \lambda_{(k+1)k} a_{k(k-1)} & \lambda_{(k+1)(k+1)} a_{kk} & \lambda_{(k+1)(k+2)} a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+2)(k-1)} a_{k(k-1)} & \lambda_{(k+2)k} a_{k(k-1)} & \lambda_{(k+2)(k+1)} a_{kk} & \lambda_{(k+2)(k+2)} a_{kk} \end{bmatrix}$$

Where,

$$\begin{cases} \sum_{j=k-1}^{k} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k+2 \\ \sum_{j=k+1}^{k+2} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k+2 \end{cases} \tag{3.6}$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A_2 within isomorphic will be:

$$\left.\begin{array}{l} \text{1- } \sum_{i=1}^{k-2} a_{ij} = 1, \quad j = 1,2,\ldots,k-2 \\ \text{2- } \sum_{j=k-1}^{k} a_{ij} = 0, \quad \forall i = 1,2,\ldots,k-2, \Leftrightarrow \\ \qquad a_{ij} = 0, \quad \forall i = 1,2,\ldots,k-2 \ \ \& \ \ \forall j = k-1,k \\ \text{3- } \sum_{i=k-1}^{k} a_{ij} = 0, \quad \forall j = 1,2,\ldots,k-2, \Leftrightarrow \\ \qquad a_{ij} = 0, \quad \forall j = 1,2,\ldots,k-2 \ \ \& \ \ \forall i = k-1,k \\ \text{4- } \sum_{i=k-1}^{k} \lambda_{ij} = \sum_{i=k+1}^{k+2} \lambda_{ij} = 1, \quad \forall j = k-1,k,k+1,k+2 \end{array}\right\} \tag{3.7}$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} R & O \\ O' & T \end{bmatrix}$$

Where,

$$\left.\begin{array}{l} \text{1) } R = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k-2)} \\ a_{21} & a_{22} & \cdots & a_{2(k-2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(k-2)1} & a_{(k-2)2} & \cdots & a_{(k-2)(k-2)} \end{bmatrix}_{(k-2)(k-2)} \\ \text{2) } O \text{ is the zero matrix of order } (k-2)\times k \\ \text{3) } T = \begin{bmatrix} \lambda_{(k-1)(k-1)} a_{(k-1)(k-1)} & \lambda_{(k-1)k} a_{(k-1)(k-1)} & \lambda_{(k-1)(k+1)} a_{(k-1)k} & \lambda_{(k-1)(k+2)} a_{(k-1)k} \\ \lambda_{k(k-1)} a_{(k-1)(k-1)} & \lambda_{kk} a_{(k-1)(k-1)} & \lambda_{k(k+1)} a_{(k-1)k} & \lambda_{k(k+2)} a_{(k-1)k} \\ \lambda_{(k+1)(k-1)} a_{k(k-1)} & \lambda_{(k+1)k} a_{k(k-1)} & \lambda_{(k+1)(k+1)} a_{kk} & \lambda_{(k+1)(k+2)} a_{kk} \\ \lambda_{(k+2)(k-1)} a_{k(k-1)} & \lambda_{(k+2)k} a_{k(k-1)} & \lambda_{(k+2)(k+1)} a_{kk} & \lambda_{(k+2)(k+2)} a_{kk} \end{bmatrix}_{(4\times 4)} \end{array}\right\} \tag{3.8}$$

3.2 The Special Case m=4, n=6

The form of the t.p.m P^* such that $\ell(P, A_2) = P^*$ is:

$$P^* = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{3.9}$$

Where, $\sum_{k=1}^{4} a_{ik} = \sum_{k=1}^{4} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{33}a_{33} & \lambda_{34}a_{33} & \lambda_{35}a_{34} & \lambda_{36}a_{34} \\ 0 & 0 & \lambda_{43}a_{33} & \lambda_{44}a_{33} & \lambda_{45}a_{34} & \lambda_{46}a_{34} \\ 0 & 0 & \lambda_{53}a_{43} & \lambda_{54}a_{43} & \lambda_{55}a_{44} & \lambda_{56}a_{44} \\ 0 & 0 & \lambda_{63}a_{43} & \lambda_{64}a_{43} & \lambda_{65}a_{44} & \lambda_{66}a_{44} \end{bmatrix}$$

4. The General Case m=2, n=k "For a certain partitions"

In this case, we have two partitions within isomorphic:

$$A_1 = \{\{1\}, \{2,3,\ldots k\}\}$$

$$A_2 = \{\{1,2\}, \{3,\ldots,k\}\}$$

Where, $k \geq 2$.

The t.p.m P^* of the lumped chain $Y(t)$ will be:

$$P^* = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.1}$$

Where, $\sum_{k=1}^{2} a_{ik} = \sum_{k=1}^{2} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k \times k)$, where $\ell(P, A_1) = P^*$ will be,

$$P = \begin{bmatrix} a_{11} & \lambda_{11}a_{12} & \lambda_{12}a_{12} & \cdots & \lambda_{1(k-1)}a_{12} \\ a_{21} & \lambda_{21}a_{22} & \lambda_{22}a_{22} & \cdots & \lambda_{2(k-1)}a_{22} \\ a_{21} & \lambda_{31}a_{22} & \lambda_{32}a_{22} & \cdots & \lambda_{3(k-1)}a_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{21} & \lambda_{k1}a_{22} & \lambda_{k2}a_{22} & \cdots & \lambda_{k(k-1)}a_{22} \end{bmatrix}, \tag{4.2}$$

Where, $\sum_{j=1}^{k-1} \lambda_{ij} = 1, \quad \forall i = 1,2,\ldots,k$

a) The partition $A_1 = \{\{1\},\{2,3,\ldots k\}\}$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A_1 within isomorphic will be:

$$\left.\begin{array}{l} \text{1- } a_{11} = 1 \Leftrightarrow a_{12} = 0 \Leftrightarrow a_{21} = 0 \Leftrightarrow a_{22} = 1 \\ \text{2- } \sum_r \lambda_{ir} = \sum_r \lambda_{rj} = 1, \quad \lambda_{ij} \geq 0, \forall i,j \end{array}\right\} \tag{4.3}$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} I & O \\ O' & T \end{bmatrix} . \tag{4.4}$$

Where,

$$\left.\begin{array}{l} \text{1) } I = [1] \\ \text{2) } O \text{ is the zero matrix of order } 1 \times (k-1) \\ \text{3) } T = \begin{bmatrix} \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2(k-1)} \\ \lambda_{31} & \lambda_{32} & \cdots & \lambda_{3(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{k(k-1)} \end{bmatrix}_{(k-1)(k-1)} \end{array}\right\} \tag{4.5}$$

b) The partition $A_2 = \{\{1,2\},\{3,\ldots,k\}\}$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k \times k)$, where $\ell(P, A_2) = P^*$ will be,

$$P = \begin{bmatrix} \lambda_{11}a_{11} & \lambda_{12}a_{11} & \lambda_{13}a_{12} & \cdots & \lambda_{1k}a_{12} \\ \lambda_{21}a_{11} & \lambda_{22}a_{11} & \lambda_{23}a_{12} & \cdots & \lambda_{2k}a_{12} \\ \lambda_{31}a_{21} & \lambda_{32}a_{21} & \lambda_{33}a_{22} & \cdots & \lambda_{3k}a_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1}a_{21} & \lambda_{k2}a_{21} & \lambda_{k3}a_{22} & \cdots & \lambda_{kk}a_{22} \end{bmatrix}, \tag{4.6}$$

Where,

$$\begin{cases} \sum_{j=1}^{2} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k \\ \sum_{j=3}^{k} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k \end{cases}$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A_2 within isomorphic will be:

$$\left.\begin{array}{l} 1\text{- } \sum_{i=1}^{2} \lambda_{ij} = 1, \quad \forall j = 1,2,\ldots,k \\ 2\text{- } \sum_{i=3}^{k} \lambda_{ij} = 1, \quad \forall j = 1,2,\ldots,k \end{array}\right\} \qquad (4.7)$$

4.1 The Special Case m=2, n=4

(i)The partition $A_1 = \{\{1\},\{2,3,4\}\}$

The form of the t.p.m P^* such that $\ell(P, A_1) = P^*$ will be the same as in (4.1). So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{31} & \lambda_{32} & \lambda_{33} \\ 0 & \lambda_{41} & \lambda_{42} & \lambda_{43} \end{bmatrix}$$

Where, $\sum_r \lambda_{ir} = \sum_r \lambda_{rj} = 1, \quad \lambda_{ij} \geq 0, \forall i, j$

(ii)The partition $A_2 = \{\{1,2\},\{3,4\}\}$

The form of the t.p.m P^* such that $\ell(P, A_2) = P^*$ will be the same as in (4.1). So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} \lambda_{11}a_{11} & \lambda_{12}a_{11} & \lambda_{13}a_{12} & \lambda_{14}a_{12} \\ \lambda_{21}a_{11} & \lambda_{22}a_{11} & \lambda_{23}a_{12} & \lambda_{24}a_{12} \\ \lambda_{31}a_{21} & \lambda_{32}a_{21} & \lambda_{33}a_{22} & \lambda_{34}a_{22} \\ \lambda_{41}a_{21} & \lambda_{42}a_{21} & \lambda_{43}a_{22} & \lambda_{44}a_{22} \end{bmatrix}$$

Where,

1- $\sum_{i=1}^{2} \lambda_{ij} = 1, \quad \forall j = 1,2,3,4$

2- $\sum_{i=3}^{4} \lambda_{ij} = 1, \quad \forall j = 1,2,3,4$

5. Summary

In this paper, we concerned with the transition probability matrix of the lumped Markov Chain $Y(t)$ when it was a double stochastic matrix. The necessary and sufficient conditions of the parameters (the values of the transition probability matrix of the original Markov Chain) have been obtained that leading to a doubly stochastic matrix of the transition probability matrix of the original Markov chain $X(t)$.

We represented the transition probability matrix P of $X(t)$ as a matrix of partition matrices with different dimensions according to the partition of the states within isomorphic. It can be shown that a part of the conditions were related to the parameters while the other parts of the conditions were depending on the probabilities of the transition probability matrix of the lumped Markov Chain $Y(t)$. Furthermore, the conditions can be derived according to the partition of the states as we have seen in section (4.1 (i)).

A possible future work plan is to discuss the necessary and the sufficient conditions of the transition probability matrix of the original Markov chain $X(t)$ to be a doubly stochastic matrix if the transition probability matrix of the weakly lumped Markov chain was doubly stochastic matrix.

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t.p.m P where $\ell(P,A)=P^*$ to be a doubly stochastic matrix for all and certain partitions within isomorphic.

2. The General Case m=k, n=k+1

In this case, we have just only one partition within isomorphic take the form:

$$A=\{\{1\},\{2\},\ldots,\{k,k+1\}\}$$

The t.p.m of the lumped chain $Y(t)$ will be,

$$P^*=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \downarrow & \downarrow & \cdots & \downarrow \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \quad (2.1)$$

Where, $\sum_k a_{ik}=\sum_k a_{kj}=1,\quad a_{ij}\geq 0,\quad \forall i,j$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k+1)\times(k+1)$, where $\ell(P,A)=P^*$ will be,

$$P=\begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1k}a_{1k} & \lambda_{1(k+1)}a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2k}a_{2k} & \lambda_{2(k+1)}a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & \lambda_{kk}a_{kk} & \lambda_{k(k+1)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)k}a_{kk} & \lambda_{(k+1)(k+1)}a_{kk} \end{bmatrix}, \quad (2.2)$$

Where, $\sum_{j=k}^{k+1}\lambda_{ij}=1,\quad \forall i=1,2,\ldots,k$

Since, the t.p.m P^* is a double stochastic matrix, then after very simple steps the necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix will be:

$$\left.\begin{array}{l} \text{1- } \sum_{i=1}^{k-1}a_{ij}=1,\quad j=1,2,\ldots,k-1 \\ \text{2- } \sum_{j=k}^{k+1}a_{ij}=0,\quad \forall i=1,2,\ldots,k-1 \Leftrightarrow a_{ij}=0,\quad \forall i=1,2,\ldots,k-1 \;\;\&\;\; \forall j=k,k+1 \\ \text{3- } \sum_{i=k}^{k+1}a_{ij}=0,\quad \forall j=1,2,\ldots,k-1 \Leftrightarrow a_{ij}=0,\quad \forall j=1,2,\ldots,k-1 \;\;\&\;\; \forall i=k,k+1 \\ \text{4- } \sum_r \lambda_{ir}=\sum_r \lambda_{rj}=1,\quad \forall i,j=k,k+1 \end{array}\right\} \quad (2.3)$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P=\begin{bmatrix} R & O \\ O' & T \end{bmatrix} \tag{2.4}$$

Where,

$$\left.\begin{array}{l} \textbf{1) } R=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k-1)} \\ a_{21} & a_{22} & \cdots & a_{2(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)(k-1)} \end{bmatrix}_{(k-1)(k-1)} \\ \textbf{2) } O \textbf{ is the zero matrix of order } (k-1)\times 2 \\ \textbf{3) } T=\begin{bmatrix} \lambda_{kk} & \lambda_{k(k+1)} \\ \lambda_{(k+1)k} & \lambda_{(k+1)(k+1)} \end{bmatrix}_{(2\times 2)} \end{array}\right\} \tag{2.5}$$

2.1 The Special Case m=4, n=5

The form of the t.p.m P^* such that $\ell(P,A)=P^*$ is:

$$P^*=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Where, $\sum_{k=1}^{4} a_{ik}=\sum_{k=1}^{4} a_{kj}=1,\ \ a_{ij}\geq 0,\ \ \forall i,j$

So, the form of all double stochastic t.p.m P will be:

$$P=\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{44} & \lambda_{45} \\ 0 & 0 & 0 & \lambda_{54} & \lambda_{55} \end{bmatrix}$$

3. The General Case m=k, n=k+2

In this case, we have only two partitions within isomorphic:

$$A_1=\{\{1\},\{2\},\ldots,\{k,k+1,k+2\}\}$$

$$A_2=\{\{1\},\{2\},\ldots,\{k-1,k\},\{k+1,k+2\}\}$$

Where, $k\geq 3$.

The t.p.m P^* of the lumped chain $Y(t)$ will take the same form as in equation (2.1).

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k+2)\times(k+2)$, where $\ell(P, A_1) = P^*$ will be,

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1k}a_{1k} & \lambda_{1(k+1)}a_{1k} & \lambda_{1(k+2)}a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2k}a_{2k} & \lambda_{2(k+1)}a_{2k} & \lambda_{2(k+2)}a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & \lambda_{kk}a_{kk} & \lambda_{k(k+1)}a_{kk} & \lambda_{k(k+2)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)k}a_{kk} & \lambda_{(k+1)(k+1)}a_{kk} & \lambda_{(k+1)(k+2)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+2)k}a_{kk} & \lambda_{(k+2)(k+1)}a_{kk} & \lambda_{(k+2)(k+2)}a_{kk} \end{bmatrix}, \qquad (3.1)$$

Where, $\sum_{j=k}^{k+2} \lambda_{ij} = 1, \quad \forall i = 1,2,\ldots,k$

a- The partition $A_1 = \{\{1\},\{2\},\ldots,\{k,k+1,k+2\}\}$

The necessary and sufficient conditions for the t.p.m P to be A double stochastic matrix with respect to the partition A_1 within isomorphic will be:

$$\left.\begin{array}{l} \text{1- } \sum_{i=1}^{k-1} a_{ij} = 1, \qquad j = 1,2,\ldots,k-1 \\ \text{2- } \sum_{j=k}^{k+2} a_{ij} = 0, \quad \forall i = 1,2,\ldots,k-1, \Leftrightarrow \\ \qquad a_{ij} = 0, \quad \forall i = 1,2,\ldots,k-1 \ \ \& \ \ \forall j = k,k+1,k+2 \\ \text{3- } \sum_{i=k}^{k+2} a_{ij} = 0, \quad \forall j = 1,2,\ldots,k-1, \Leftrightarrow \\ \qquad a_{ij} = 0, \quad \forall j = 1,2,\ldots,k-1 \ \ \& \ \ \forall i = k,k+1,k+2 \\ \text{4- } \sum_r \lambda_{ir} = \sum_r \lambda_{rj} = 1, \quad \forall i,j = k,k+1,k+2 \end{array}\right\} \qquad (3.2)$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} R & O \\ O' & T \end{bmatrix} \qquad (3.3)$$

Where,

$$\left.\begin{array}{l}
\text{1) } R \text{ is the same matrix as in (2.5).} \\
\text{2) } O \text{ is the zero matrix of order } (k-1)\times 3 \\
\text{3) } T = \begin{bmatrix} \lambda_{kk} & \lambda_{k(k+1)} & \lambda_{k(k+2)} \\ \lambda_{(k+1)k} & \lambda_{(k+1)(k+1)} & \lambda_{(k+1)(k+2)} \\ \lambda_{(k+2)k} & \lambda_{(k+2)(k+1)} & \lambda_{(k+2)(k+2)} \end{bmatrix}_{(3\times 3)}
\end{array}\right\} \qquad (3.4)$$

3.1 The Special Case m=3, n=5

The form of the t.p.m P^* such that $\ell(P, A_1) = P^*$ is:

$$P^* = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (3.5)$$

Where, $\sum_{k=1}^{3} a_{ik} = \sum_{k=1}^{3} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{33} & \lambda_{34} & \lambda_{35} \\ 0 & 0 & \lambda_{43} & \lambda_{44} & \lambda_{45} \\ 0 & 0 & \lambda_{53} & \lambda_{54} & \lambda_{55} \end{bmatrix}$$

b- The partition $A_2 = \{\{1\}, \{2\}, \ldots, \{k-1, k\}, \{k+1, k+2\}\}$

The form of all transition matrices P of the **Markov Chain** $X(t)$ **of** order $(k+2)\times(k+2)$, where $\ell(P, A_2) = P^*$ will be,

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & \lambda_{1(k-1)}a_{1(k-1)} & \lambda_{1k}a_{1(k-1)} & \lambda_{1(k+1)}a_{1k} & \lambda_{1(k+2)}a_{1k} \\ a_{21} & a_{22} & \cdots & \lambda_{2(k-1)}a_{2(k-1)} & \lambda_{2k}a_{2(k-1)} & \lambda_{2(k+1)}a_{2k} & \lambda_{2(k+2)}a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & \lambda_{(k-1)(k-1)}a_{(k-1)(k-1)} & \lambda_{(k-1)k}a_{(k-1)(k-1)} & \lambda_{(k-1)(k+1)}a_{(k-1)k} & \lambda_{(k-1)(k+2)}a_{(k-1)k} \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & \lambda_{k(k-1)}a_{(k-1)(k-1)} & \lambda_{kk}a_{(k-1)(k-1)} & \lambda_{k(k+1)}a_{(k-1)k} & \lambda_{k(k+2)}a_{(k-1)k} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+1)(k-1)}a_{k(k-1)} & \lambda_{(k+1)k}a_{k(k-1)} & \lambda_{(k+1)(k+1)}a_{kk} & \lambda_{(k+1)(k+2)}a_{kk} \\ a_{k1} & a_{k2} & \cdots & \lambda_{(k+2)(k-1)}a_{k(k-1)} & \lambda_{(k+2)k}a_{k(k-1)} & \lambda_{(k+2)(k+1)}a_{kk} & \lambda_{(k+2)(k+2)}a_{kk} \end{bmatrix}$$

Where,

$$\begin{cases} \sum_{j=k-1}^{k} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k+2 \\ \sum_{j=k+1}^{k+2} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k+2 \end{cases} \tag{3.6}$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A_2 within isomorphic will be:

$$\left.\begin{array}{l}
\text{1- } \sum_{i=1}^{k-2} a_{ij} = 1, \quad j = 1,2,\ldots,k-2 \\
\text{2- } \sum_{j=k-1}^{k} a_{ij} = 0, \quad \forall i = 1,2,\ldots,k-2, \Leftrightarrow \\
\qquad a_{ij} = 0, \quad \forall i = 1,2,\ldots,k-2 \ \& \ \forall j = k-1,k \\
\text{3- } \sum_{i=k-1}^{k} a_{ij} = 0, \quad \forall j = 1,2,\ldots,k-2, \Leftrightarrow \\
\qquad a_{ij} = 0, \quad \forall j = 1,2,\ldots,k-2 \ \& \ \forall i = k-1,k \\
\text{4- } \sum_{i=k-1}^{k} \lambda_{ij} = \sum_{i=k+1}^{k+2} \lambda_{ij} = 1, \quad \forall j = k-1,k,k+1,k+2
\end{array}\right\} \tag{3.7}$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} R & O \\ O' & T \end{bmatrix}$$

Where,

$$\left.\begin{array}{l}
\text{1) } R = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k-2)} \\ a_{21} & a_{22} & \cdots & a_{2(k-2)} \\ \cdots & \cdots & \cdots & \cdots \\ a_{(k-2)1} & a_{(k-2)2} & \cdots & a_{(k-2)(k-2)} \end{bmatrix}_{(k-2)(k-2)} \\
\text{2) } O \text{ is the zero matrix of order } (k-2)\times k \\
\text{3) } T = \begin{bmatrix} \lambda_{(k-1)(k-1)} a_{(k-1)(k-1)} & \lambda_{(k-1)k} a_{(k-1)(k-1)} & \lambda_{(k-1)(k+1)} a_{(k-1)k} & \lambda_{(k-1)(k+2)} a_{(k-1)k} \\ \lambda_{k(k-1)} a_{(k-1)(k-1)} & \lambda_{kk} a_{(k-1)(k-1)} & \lambda_{k(k+1)} a_{(k-1)k} & \lambda_{k(k+2)} a_{(k-1)k} \\ \lambda_{(k+1)(k-1)} a_{k(k-1)} & \lambda_{(k+1)k} a_{k(k-1)} & \lambda_{(k+1)(k+1)} a_{kk} & \lambda_{(k+1)(k+2)} a_{kk} \\ \lambda_{(k+2)(k-1)} a_{k(k-1)} & \lambda_{(k+2)k} a_{k(k-1)} & \lambda_{(k+2)(k+1)} a_{kk} & \lambda_{(k+2)(k+2)} a_{kk} \end{bmatrix}_{(4\times 4)}
\end{array}\right\} \tag{3.8}$$

3.2 The Special Case m=4, n=6

The form of the t.p.m P^* such that $\ell(P, A_2) = P^*$ is:

$$P^* = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{3.9}$$

Where, $\sum_{k=1}^{4} a_{ik} = \sum_{k=1}^{4} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{33}a_{33} & \lambda_{34}a_{33} & \lambda_{35}a_{34} & \lambda_{36}a_{34} \\ 0 & 0 & \lambda_{43}a_{33} & \lambda_{44}a_{33} & \lambda_{45}a_{34} & \lambda_{46}a_{34} \\ 0 & 0 & \lambda_{53}a_{43} & \lambda_{54}a_{43} & \lambda_{55}a_{44} & \lambda_{56}a_{44} \\ 0 & 0 & \lambda_{63}a_{43} & \lambda_{64}a_{43} & \lambda_{65}a_{44} & \lambda_{66}a_{44} \end{bmatrix}$$

4. The General Case m=2, n=k "For a certain partitions"

In this case, we have two partitions within isomorphic:

$$A_1 = \{\{1\}, \{2,3,\dots k\}\}$$

$$A_2 = \{\{1,2\}, \{3,\dots,k\}\}$$

Where, $k \geq 2$.

The t.p.m P^* of the lumped chain $Y(t)$ will be:

$$P^* = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4.1}$$

Where, $\sum_{k=1}^{2} a_{ik} = \sum_{k=1}^{2} a_{kj} = 1, \quad a_{ij} \geq 0, \quad \forall i, j$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k \times k)$, where $\ell(P, A_1) = P^*$ will be,

$$P = \begin{bmatrix} a_{11} & \lambda_{11}a_{12} & \lambda_{12}a_{12} & \cdots & \lambda_{1(k-1)}a_{12} \\ a_{21} & \lambda_{21}a_{22} & \lambda_{22}a_{22} & \cdots & \lambda_{2(k-1)}a_{22} \\ a_{21} & \lambda_{31}a_{22} & \lambda_{32}a_{22} & \cdots & \lambda_{3(k-1)}a_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{21} & \lambda_{k1}a_{22} & \lambda_{k2}a_{22} & \cdots & \lambda_{k(k-1)}a_{22} \end{bmatrix}, \tag{4.2}$$

Where, $\sum_{j=1}^{k-1} \lambda_{ij} = 1, \quad \forall i = 1,2,\ldots,k$

a) The partition $A_1 = \{\{1\},\{2,3,\ldots k\}\}$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A_1 within isomorphic will be:

$$\left.\begin{array}{l} \text{1- } a_{11} = 1 \Leftrightarrow a_{12} = 0 \Leftrightarrow a_{21} = 0 \Leftrightarrow a_{22} = 1 \\ \text{2- } \sum_r \lambda_{ir} = \sum_r \lambda_{rj} = 1, \quad \lambda_{ij} \geq 0, \forall i,j \end{array}\right\} \qquad (4.3)$$

Therefore, the form of all t.p.m P where P is a double stochastic matrix can be written in the form:

$$P = \begin{bmatrix} I & O \\ O' & T \end{bmatrix}. \qquad (4.4)$$

Where,

$$\left.\begin{array}{l} \text{1) } I = [1] \\ \text{2) } O \text{ is the zero matrix of order } 1 \times (k-1) \\ \text{3) } T = \begin{bmatrix} \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2(k-1)} \\ \lambda_{31} & \lambda_{32} & \cdots & \lambda_{3(k-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{k(k-1)} \end{bmatrix}_{(k-1)(k-1)} \end{array}\right\} \qquad (4.5)$$

b) The partition $A_2 = \{\{1,2\},\{3,\ldots,k\}\}$

The form of all transition matrices P of the Markov Chain $X(t)$ of order $(k \times k)$, where $\ell(P, A_2) = P^*$ will be,

$$P = \begin{bmatrix} \lambda_{11}a_{11} & \lambda_{12}a_{11} & \lambda_{13}a_{12} & \cdots & \lambda_{1k}a_{12} \\ \lambda_{21}a_{11} & \lambda_{22}a_{11} & \lambda_{23}a_{12} & \cdots & \lambda_{2k}a_{12} \\ \lambda_{31}a_{21} & \lambda_{32}a_{21} & \lambda_{33}a_{22} & \cdots & \lambda_{3k}a_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1}a_{21} & \lambda_{k2}a_{21} & \lambda_{k3}a_{22} & \cdots & \lambda_{kk}a_{22} \end{bmatrix}, \qquad (4.6)$$

Where,

$$\begin{cases} \sum_{j=1}^{2} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k \\ \sum_{j=3}^{k} \lambda_{ij} = 1, & \forall i = 1,2,\ldots,k \end{cases}$$

The necessary and sufficient conditions for the t.p.m P to be a double stochastic matrix with respect to the partition A_2 within isomorphic will be:

$$\left.\begin{array}{l} \text{1- } \sum_{i=1}^{2} \lambda_{ij} = 1, \quad \forall j = 1,2,\ldots,k \\ \\ \text{2- } \sum_{i=3}^{k} \lambda_{ij} = 1, \quad \forall j = 1,2,\ldots,k \end{array}\right\} \qquad (4.7)$$

4.1 The Special Case m=2, n=4

(i) The partition $A_1 = \{\{1\},\{2,3,4\}\}$

The form of the t.p.m P^* such that $\ell(P, A_1) = P^*$ will be the same as in (4.1). So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{31} & \lambda_{32} & \lambda_{33} \\ 0 & \lambda_{41} & \lambda_{42} & \lambda_{43} \end{bmatrix}$$

Where, $\sum_r \lambda_{ir} = \sum_r \lambda_{rj} = 1, \quad \lambda_{ij} \geq 0, \forall i, j$

(ii) The partition $A_2 = \{\{1,2\},\{3,4\}\}$

The form of the t.p.m P^* such that $\ell(P, A_2) = P^*$ will be the same as in (4.1). So, the form of all double stochastic t.p.m P will be:

$$P = \begin{bmatrix} \lambda_{11}a_{11} & \lambda_{12}a_{11} & \lambda_{13}a_{12} & \lambda_{14}a_{12} \\ \lambda_{21}a_{11} & \lambda_{22}a_{11} & \lambda_{23}a_{12} & \lambda_{24}a_{12} \\ \lambda_{31}a_{21} & \lambda_{32}a_{21} & \lambda_{33}a_{22} & \lambda_{34}a_{22} \\ \lambda_{41}a_{21} & \lambda_{42}a_{21} & \lambda_{43}a_{22} & \lambda_{44}a_{22} \end{bmatrix}$$

Where,

1- $\sum_{i=1}^{2} \lambda_{ij} = 1, \quad \forall j = 1,2,3,4$

2- $\sum_{i=3}^{4} \lambda_{ij} = 1, \quad \forall j = 1,2,3,4$

5. Summary

In this paper, we concerned with the transition probability matrix of the lumped Markov Chain $Y(t)$ when it was a double stochastic matrix. The necessary and sufficient conditions of the parameters (the values of the transition probability matrix of the original Markov Chain) have been obtained that leading to a doubly stochastic matrix of the transition probability matrix of the original Markov chain $X(t)$.

We represented the transition probability matrix P of $X(t)$ as a matrix of partition matrices with different dimensions according to the partition of the states within isomorphic. It can be shown that a part of the conditions were related to the parameters while the other parts of the conditions were depending on the probabilities of the transition probability matrix of the lumped Markov Chain $Y(t)$. Furthermore, the conditions can be derived according to the partition of the states as we have seen in section (4.1 (i)).

A possible future work plan is to discuss the necessary and the sufficient conditions of the transition probability matrix of the original Markov chain $X(t)$ to be a doubly stochastic matrix if the transition probability matrix of the weakly lumped Markov chain was doubly stochastic matrix.

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