Unit Root Test of Bounded AR (2) Model with Constant and with Independent Errors

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Abstract

In this paper, unit root test of bounded AR (2) model with constant and with independent errors has been derived, where estimation of the model, asymptotic distributions of OLS estimators under different tests of hypothesis and asymptotic distributions of the $t-type$ statistics under different tests of hypothesis have been derived. Also, the simulation results of the bias, mean squared error (MSE), Thiel's inequality coefficient (Thiel's U) and power of the test for OLS estimators of bounded AR (2) model with constant and with independent errors approved the alternative hypothsis H_a more than the null hypothsis H_0.

Keywords: Bounded AR (2) model, asymptotic distributions, OLS estimators, tests of hypothesis, the $t-type$ statistics, mean squared error, Thiel's inequality coefficient and power of the test.

1. Introduction

Many unit root tests have been developed for testing the null hypothesis of a unit root against the alternative of stationarity; the tests for unit roots in AR (1) processes were first proposed and investigated by Dickey and Fuller (1979, 1981) but these unit root tests are proposed to unbounded time series in case of independent error terms.

Cavaliere (2000) tested the presence of unknown boundaries which constrain the sample path to lie within a closed interval that is in the framework of integrated processes of AR (1) model with a unit root or random walk model (with and without linear trend). In (2002), he introduced the logged nominal exchange rates $\{y_t\}$ that change in time accordingly to a first-order integrated process, $I(1)$ within the framework of non-managed flexible exchange rates. Then in (2005), he developed an asymptotic theory for integrated and near-integrated time series whose range is constrained in some ways. Such a framework arises when integration and cointegration analysis are applied to persistent series which are bounded either by construction or because they are subject to control.

Cavaliere and Xu (2011) defined bounded process as time series x_t with (fixed) bounds at $\underline{b}, \overline{b}$; $\underline{b} < \overline{b}$, is a stochastic process satisfying $x_t \in [\underline{b}, \overline{b}]$ for all t.

Carrion and Gadea (2013) showed that the use of generalized least squares (GLS) detrending procedures leads to important empirical power gains compared to ordinary least squares (OLS) detrending method when testing the null hypothesis of unit root for bounded processes. In (2015), they discussed the unit root testing when the range of the time series is bounded considering the presence of multiple structural breaks.

Form the previous researches it can be notice that the concentration was on the model of bounded AR (1) with constant (without constant) under various assumptions for the error terms, and in this paper the concentration will be on the model of bounded AR (2) with constant in the case of independent errors.

2. Unit Root Test of Bounded AR (2) Model with Constant

The bounded second order autoregressive AR (2) model with constant takes the following form:

$$y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + e_t, \qquad t = 1,\dots,T. \tag{1}$$

Under the following assumptions:

1. y_t is bounded time series with fixed bounds with lower bound at $\underline{b}$ and upper bound at $\overline{b}$ i.e. $y_t \in [\underline{b}, \overline{b}]$ and $y_0 = y_{-1} = 0$,
2. $\underline{b} = \underline{c}\, T^{1/2}$, $\overline{b} = \overline{c}\, T^{1/2}$, T is the sample size, $\underline{c}, \overline{c} \in R/\{0\}$ and $\underline{c} < \overline{c}$,
3. $e_t \sim$ IID $N(0, \sigma^2)$,
4. ρ_1, ρ_2 are the autoregressive coefficients and α is the constant term.

2.1 Estimation of the Model

Equation (1) can be rewritten in matrix form as follows:

$$Y = X\beta + \mathbf{e} \tag{2}$$

Where:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix},\ X = \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix},\ \beta = \begin{bmatrix} \alpha \\ \rho_1 \\ \rho_2 \end{bmatrix} \text{ and } \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_T \end{bmatrix}$$

Then, under the assumptions of model (1) the OLS estimators of parameters of model (1) can be obtained as follows:

$$\hat{\beta} = (X'X)^{-1} X'Y$$

Then, the OLS estimators $\hat{\alpha}, \hat{\rho}_1$ and $\hat{\rho}_2$ will be as follows:

$$\hat{\beta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\rho}_1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-2} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 & \sum_{t=1}^{T} y_{t-1} y_{t-2} \\ \sum_{t=1}^{T} y_{t-2} & \sum_{t=1}^{T} y_{t-1} y_{t-2} & \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{T} y_t \\ \sum_{t=1}^{T} y_t y_{t-1} \\ \sum_{t=1}^{T} y_t y_{t-2} \end{pmatrix}$$

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\rho}_1\\ \hat{\rho}_2\end{pmatrix}=D^{-1}\begin{pmatrix}L_{11} & L_{12} & L_{13}\\ L_{21} & L_{22} & L_{23}\\ L_{31} & L_{32} & L_{33}\end{pmatrix}\begin{pmatrix}M_{11}\\ M_{12}\\ M_{13}\end{pmatrix}$$

$$\hat{\alpha}=\frac{L_{11}M_{11}+L_{12}M_{12}+L_{13}M_{13}}{D} \quad , \quad \hat{\rho}_1=\frac{L_{21}M_{11}+L_{22}M_{12}+L_{23}M_{13}}{D}$$

$$\hat{\rho}_2=\frac{L_{31}M_{11}+L_{32}M_{12}+L_{33}M_{13}}{D}$$

Where:

$$D=\det(X'X)^{-1},\ L_{11}=[\Sigma_{t=1}^{T} y_{t-1}^2][\Sigma_{t=1}^{T} y_{t-2}^2]-[\Sigma_{t=1}^{T} y_{t-1}y_{t-2}]^2$$
$$L_{22}=T\Sigma_{t=1}^{T} y_{t-2}^2-[\Sigma_{t=1}^{T} y_{t-2}]^2\ ,\ L_{33}=T\Sigma_{t=1}^{T} y_{t-1}^2-[\Sigma_{t=1}^{T} y_{t-1}]^2$$
$$L_{12}=L_{21}=[\Sigma_{t=1}^{T} y_{t-2}][\Sigma_{t=1}^{T} y_{t-1}y_{t-2}]-[\Sigma_{t=1}^{T} y_{t-1}][\Sigma_{t=1}^{T} y_{t-2}^2]$$
$$L_{13}=L_{31}=[\Sigma_{t=1}^{T} y_{t-1}][\Sigma_{t=1}^{T} y_{t-1}y_{t-2}]-[\Sigma_{t=1}^{T} y_{t-2}][\Sigma_{t=1}^{T} y_{t-1}^2]$$
$$L_{23}=L_{32}=[\Sigma_{t=1}^{T} y_{t-1}][\Sigma_{t=1}^{T} y_{t-2}]-T\Sigma_{t=1}^{T} y_{t-1}y_{t-2}$$
$$M_{11}=\Sigma_{t=1}^{T} y_t\ ;\ M_{12}=\Sigma_{t=1}^{T} y_t y_{t-1}\ ,M_{13}=\Sigma_{t=1}^{T} y_t y_{t-2}$$

2.2 Asymptotic Distributions of OLS Estimators under Different Tests of Hypothesis

Concepts of relative magnitude or order of magnitude are useful in investigating limiting behavior of random variables, where if $h(x)$ and $g(x)$ are two real functions that have a common domain $D\subset R$, and if the following relationship is exists for any positive constant $k(k>0)$

$$\lim_{x\to x_0}\left|\frac{h(x)}{g(x)}\right|\le k \quad , \quad x\in(D-x_0)$$

Then:

$$h(x)=O(g(x)). \tag{3}$$

Schatzman (2002)

which means that $h(x)$ is at most of order $g(x)$.

One of the uses of Brownian Motion process is the possibility to obtain a more general formulation of the central limit theorem, where the simplest formulation of the central limit theorem is if $e_t \sim \text{IID}N(0,\sigma^2)$, then the sample mean $\bar{e}_T$ of these random variables achieved the following asymptotic distribution:

$$\sqrt{T}\,\bar{e}_T=\sqrt{T}\,\sum_{t=1}^{T}e_t/T=\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}e_t\xrightarrow{d}N(0,\sigma^2) \quad (4)$$

Amer (2015)

And if the sequence $e_t\sim \text{IID}N(0,\sigma^2)$, $X_T(r)=\frac{1}{T}\sum_{t=1}^{[T_r]^*}e_t$, where the random variable $X_T(r)$ represents the sample mean that calculated from the first (r) ratio of observation $r=\frac{[T_r]^*}{T}$, $[T_r]^*=0,1,\ldots,T$ and $r\in[0,1]$, then the asymptotic distribution of $\sqrt{T}X_T(r)/\sigma$ is as follows:

$$\sqrt{T}X_T(r)/\sigma\xrightarrow{d}W(r) \quad (5)$$

Where $W(r)$ is a Standard Brownian Motion process, $r\in[0,1]$ and when $r=1$ then $W(1)\sim N(0,1)$, and $\sqrt{T}X_T(.)/\sigma$, $T=1,2,\ldots$ is a sequence of stochastic functions that have an asymptotic distribution that can be described by Standard Brownian Motion $W(\cdot)$ as follows:

$$\sqrt{T}X_T(\cdot)/\sigma\xrightarrow{d}W(\cdot) \quad (6)$$

The result in equation (6) is called the Functional Central Limit Theorem (FCLT) for independent errors, where $X_T(\cdot)$ is a stochastic function and $X_T(r)$ is the value of this stochastic function at time r i.e. $X_T(\cdot)$ is a function and $X_T(r)$ is a variable, when $r=1$ then the function in equation (5) is as follows:

$$\sqrt{T}X_T(1)=\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T}e_t\xrightarrow{d}\sigma W(1)\sim\sigma N(0,1)=N(0,\sigma^2) \quad (7)$$

Davidson (1994)

If y_t is a pure random walk without drift as $y_t=y_{t-1}+e_t$, where $e_t\sim \text{IID}N(0,\sigma^2)$, $y_0=y_{-1}=0$, and when y_t is bounded time series with fixed bounds at $\underline{b},\bar{b}$; $\underline{b}<\bar{b}$, i.e. $y_t\in[\underline{b},\bar{b}]$, $\underline{b}=\underline{c}\,T^{1/2}$, $\bar{b}=\bar{c}\,T^{1/2}$ then the Standard Brownian Motion in equation (5) will be replaced by the Regulated Brownian Motion $W_{\underline{c}}^{\bar{c}}(r)$ with boundaries at $\underline{c}$, $\bar{c}$ and $r\in[0,1]$ as follows:

$$\left.\begin{array}{l} 1)\ \sqrt{T}X_T(r)=\frac{1}{\sqrt{T}}\sum_{t=1}^{[T_r]^*}e_t\xrightarrow{d}\sigma\,W_{\underline{c}}^{\bar{c}}(r)\\ \text{and at } (r=1)\\ 2)\ \sqrt{T}X_T(1)=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_t\xrightarrow{d}\sigma\,W_{\underline{c}}^{\bar{c}}(1)\end{array}\right\} \quad (8)$$

Cavaliere (2005)

Phillips (1986, 1987) used the Functional Central Limit Theorem to derive the asymptotic distributions for the statistics that are based on the unit root processes in the random walk process without drift as follows:

If y_t is a pure random walk without drift as $y_t = y_{t-1} + e_t$, where $e_t \sim \text{IID}N(0,\sigma^2)$, $y_0 = y_{-1} = 0$, based on Phillips (1986, 1987) and using equation (8) then, the following results are obtained:

$$\left.\begin{array}{l} 1)\ T^{-3/2}\sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \sigma \int_0^1 W(r)_{\underline{c}}^{\bar{c}}\, dr \\ 2)\ T^{-1}\sum_{t=1}^{T} y_{t-1}\, e_t \xrightarrow{d} \frac{1}{2}\sigma^2 \{ [W(1)_{\underline{c}}^{\bar{c}}]^2 - 1 \} \\ 3)\ T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{d} \sigma^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2\, dr \end{array}\right\} \quad (9)$$

If the results of equations (8) and (9) in case of independent errors are hold and by using equation (3) then the results for the orders of convergence of estimators in these equations will be as follows:

$$\left.\begin{array}{l} 1)\ T = O_p(T) \\ 2)\ \sum_{t=1}^{T} e_t = O_p(T^{1/2}) \\ 3)\ \sum_{t=1}^{T} y_{t-1} = O_p(T^{3/2}) \\ 4)\ \sum_{t=1}^{T} y_{t-1}^2 = O_p(T^2) \\ 5)\ \sum_{t=1}^{T} y_{t-1}\, e_t = O_p(T) \\ 6)\ \sum_{t=1}^{T} e_t^2 = O_p(T) \end{array}\right\} \quad (10)$$

Amer (2015)

The asymptotic distributions of OLS estimators $\hat{\alpha}$, $\hat{\rho}_1$ and $\hat{\rho}_2$ for bounded AR (2) model that represented by equation (1) under the test $H_0 : \alpha = 0,\ \rho_1 = 1,\ \rho_2 = 0$, (i.e. $y_t = y_{t-1} + e_t$) against $H_a : \alpha \neq 0,\ |\rho_1| < 1,\ |\rho_2| < 1$, (i.e. $y_t = \alpha + \rho_1\, y_{t-1} + \rho_2\, y_{t-2} + e_t$) will be derived as follows:

Lemma (1): If y_t is a pure random walk without drift as $y_t = y_{t-1} + e_t$ and has the same assumptions of equation (8) then as $T \to \infty$ the following results are obtained:

$$\left.\begin{array}{l} 1)\ T^{-3/2}\sum_{t=1}^{T} y_{t-2} \xrightarrow{d} \sigma \int_0^1 W_{\underline{c}}^{\bar{c}}(r)\, dr \\ 2)\ T^{-1}\sum_{t=1}^{T} y_{t-2}\, e_t \xrightarrow{d} \frac{1}{2}\sigma^2 \{ [W_{\underline{c}}^{\bar{c}}(1)]^2 - 1 \} \\ 3)\ T^{-2}\sum_{t=1}^{T} y_{t-2}^2 \xrightarrow{d} \sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr \\ 4)\ T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2} \xrightarrow{d} \sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr \end{array}\right\} \quad (11)$$

Proof:

Part (1)

From the successive substituting of y_t then:

$$y_{t-2} = y_{t-1} - e_{t-1} \tag{12}$$

From equation (12) then:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-2} = T^{-3/2}\sum_{t=1}^{T} y_{t-1} - T^{-3/2}\sum_{t=1}^{T} e_{t-1} \tag{13}$$

By using equation (8 (2)) then:

$$T^{-3/2}\sum_{t=1}^{T} e_{t-1} \xrightarrow{d} 0 \tag{14}$$

By using equation (9 (1)) then:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \sigma\int_0^1 W_{\underline{c}}^{\bar{c}}(r)\,dr \tag{15}$$

Then, by substituting from equations (14) and (15) in equation (13) it can be concluded that:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-2} \xrightarrow{d} \sigma\int_0^1 W_{\underline{c}}^{\bar{c}}(r)\,dr \quad , \quad (y_{-1}=0)$$

Part (2)

$$T^{-1}\sum_{t=1}^{T} y_{t-2}\, e_t = T^{-1}\sum_{t=1}^{T} y_{t-1} e_t - T^{-1}\sum_{t=1}^{T} e_{t-1} e_t \tag{16}$$

Since, $e_t \sim \text{IID}N(0,\sigma^2)$ then:

$$T^{-1}\sum_{t=1}^{T} e_{t-1} e_t \xrightarrow{p} 0 \tag{17}$$

By using equation (9 (2)) then:

$$T^{-1}\sum_{t=1}^{T} y_{t-1}\, e_t \xrightarrow{d} \tfrac{1}{2}\sigma^2\{[W_{\underline{c}}^{\bar{c}}(1)]^2 - 1\} \tag{18}$$

Then, by substituting from (17) and (18) in equation (16) it can be concluded that:

$$T^{-1}\sum_{t=1}^{T} y_{t-2}\, e_t \xrightarrow{d} \tfrac{1}{2}\sigma^2\{[W_{\underline{c}}^{\bar{c}}(1)]^2 - 1\} \ , \ (y_{-1}=0)$$

Part (3)

$$T^{-2}\sum_{t=1}^{T} y_{t-2}^2 = T^{-2}\sum_{t=1}^{T} y_{t-1}^2 - 2T^{-2}\sum_{t=1}^{T} y_{t-1} e_{t-1} + T^{-2}\sum_{t=1}^{T} e_{t-1}^2 \tag{19}$$

From equation (10) the order of convergence of $\sum_{t=1}^{T} y_{t-1}^2 = O_p(T^2)$ and the order of convergence of $\sum_{t=1}^{T} e_t^2 = O_p(T)$ then:

$$\left.\begin{array}{l} 1) T^{-2}\sum_{t=1}^{T} y_{t-1} e_{t-1} \xrightarrow{d} 0 \\ 2) T^{-2}\sum_{t=1}^{T} e_{t-1}^{2} \xrightarrow{d} 0 \end{array}\right\} \quad (20)$$

By using equation (9 (3)) then:

$$T^{-2}\sum_{t=1}^{T} y_{t-1}^{2} \xrightarrow{d} \sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}\,dr \quad (21)$$

Then, by substituting from equations (20) and (21) in equation (19) it can be concluded that:

$$T^{-2}\sum_{t=1}^{T} y_{t-2}^{2} \xrightarrow{d} \sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}\,dr \quad , (y_{-1}=0)$$

Part (4)

$$T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2} = T^{-2}\sum_{t=1}^{T} y_{t-1}^{2} - T^{-2}\sum_{t=1}^{T} y_{t-1}\, e_{t-1} \quad (22)$$

By using equation (9 (3)) then:

$$T^{-2}\sum_{t=1}^{T} y_{t-1}^{2} \xrightarrow{d} \sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}\,dr \quad (23)$$

Then, by substituting from equations (20 (1)) and (23) in equation (22) it can be concluded that:

$$T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2} \xrightarrow{d} \sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}\,dr$$

Lemma (2): For model (1) and under the test $H_0:\alpha=0, \rho_1=1, \rho_2=0$, the asymptotic distributions of $T^{1/2}\hat{\alpha}$, $T(\hat{\rho}_1-1)$ and $T\hat{\rho}_2$ will be as follows:

$$1) T^{1/2}\hat{\alpha} \xrightarrow{d} \frac{[\sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}dr][\sigma \bar{W_{\underline{c}}^{c}}(1)] - [\sigma\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]\,dr][\frac{1}{2}\sigma^{2}\{[\bar{W_{\underline{c}}^{c}}(1)]^{2}-1\}]}{\sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}dr - \left\{\sigma\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]\,dr\right\}^{2}}$$

$$2) T(\hat{\rho}_1-1) \xrightarrow{d} \frac{[-\sigma\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]\,dr][\sigma \bar{W_{\underline{c}}^{c}}(1)] + \frac{1}{2}\sigma^{2}\{[\bar{W_{\underline{c}}^{c}}(1)]^{2}-1\}}{\sigma^{2}\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]^{2}dr - \left\{\sigma\int_{0}^{1}[\bar{W_{\underline{c}}^{c}}(r)]\,dr\right\}^{2}} - z_3$$

$$3) T\hat{\rho}_2 \xrightarrow{d} z_3 , \; z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}$$

Proof:

Since:

$$\hat{\beta} = (X'X)^{-1}X'Y$$

By using equation (2) then:

$$\hat{\beta} - \beta = (X'X)^{-1}X'\mathbf{e}$$

Under the null hypothesis that $H_0:\alpha=0, \rho_1=1, \rho_2=0$ or $\beta'=(0 \quad 1 \quad 0)$ then:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\rho}_1 - 1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-2} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 & \sum_{t=1}^{T} y_{t-1}y_{t-2} \\ \sum_{t=1}^{T} y_{t-2} & \sum_{t=1}^{T} y_{t-1}y_{t-2} & \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{T} e_t \\ \sum_{t=1}^{T} y_{t-1} e_t \\ \sum_{t=1}^{T} y_{t-2} e_t \end{pmatrix} \quad (24)$$

From equation (10), the order of convergence of T, $\sum_{t=1}^{T} e_t$, $\sum_{t=1}^{T} y_{t-1} (\sum_{t=2}^{T} y_{t-2})$, $\sum_{t=1}^{T} y_{t-1}^2 (\sum_{t=2}^{T} y_{t-2}^2)$ and $\sum_{t=1}^{T} y_{t-1} e_t$ will be $O_p(T)$, $O_p(T^{1/2})$, $O_p(T^{3/2})$, $O_p(T^2)$ and $O_p(T)$ respectively. Also, from equation (11 (2)) and by using the role of equation (3) then the order of convergence of $\sum_{t=1}^{T} y_{t-2} e_t = O_p(T)$, and from equation (11 (4)) and by using the role of equation (3) then the order of convergence of $\sum_{t=1}^{T} y_{t-1} y_{t-2} = O_p(T^2)$.

Then, the order of convergence of the elements in equation (24) will be as follows:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\rho}_1 - 1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} O_p(T) & O_p(T^{3/2}) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) & O_p(T^2) \\ O_p(T^{3/2}) & O_p(T^2) & O_p(T^2) \end{pmatrix}^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T) \\ O_p(T) \end{pmatrix}$$

Then, to obtain the asymptotic distributions of the estimators equation (24) will be multiplied by the following scaling matrix:

$$\psi_T = \begin{pmatrix} T^{1/2} & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{pmatrix}$$

Then, equation (24) will be as follows:

$$\psi_T (\hat{\beta} - \beta) = \{\psi_T^{-1} (X'X) \psi_T^{-1}\}^{-1} \psi_T^{-1} X'\mathbf{e}$$

$$\begin{pmatrix} T^{1/2} \hat{\alpha} \\ T(\hat{\rho}_1 - 1) \\ T \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} 1 & T^{-3/2} \sum_{t=1}^{T} y_{t-1} & T^{-3/2} \sum_{t=1}^{T} y_{t-2} \\ T^{-3/2} \sum_{t=1}^{T} y_{t-1} & T^{-2} \sum_{t=1}^{T} y_{t-1}^2 & T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} \\ T^{-3/2} \sum_{t=1}^{T} y_{t-2} & T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} & T^{-2} \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} T^{-1/2} \sum_{t=1}^{T} e_t \\ T^{-1} \sum_{t=1}^{T} y_{t-1} e_t \\ T^{-1} \sum_{t=1}^{T} y_{t-2} e_t \end{pmatrix} \quad (25)$$

From equation (8 (2)), $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} e_t \xrightarrow{d} \sigma W_{\underline{c}}^{\bar{c}}(1)$, from equation (9), $T^{-3/2} \sum_{t=1}^{T} y_{t-1}$, $T^{-1} \sum_{t=1}^{T} y_{t-1} e_t$ and $T^{-2} \sum_{t=1}^{T} y_{t-1}^2$ convergence in distribution to $\sigma \int_0^1 W(r)_{\underline{c}}^{\bar{c}} dr$, $\frac{1}{2}\sigma^2 \{ [W(1)_{\underline{c}}^{\bar{c}}]^2 - 1\}$ and $\sigma^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2 dr$ respectively.

Also, from equation (11), $T^{-3/2}\sum_{t=1}^{T} y_{t-2}$, $T^{-1}\sum_{t=1}^{T} y_{t-2}\, e_t$ and $T^{-2}\sum_{t=1}^{T} y_{t-2}^{2} (T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2})$ convergence in distribution to $\sigma\int_0^1 W_{\underline{c}}^{\bar{c}}(r)\,dr$, $\frac{1}{2}\sigma^2\{[W_{\underline{c}}^{\bar{c}}(1)]^2-1\}$ and $\sigma^2\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr$ respectively.

Then, as $T\to\infty$ and by using the above results equation (25) will be as follows:

$$x_1 = A_1^{-1}\, h_1 \ ,\ x_1 \in R_{(3\times 1)} \text{ (i.e. vector of order } (3\times 1) \text{ of real numbers)} \qquad (26)$$

Where:

$$x_1 = \lim_{T\to\infty}\begin{pmatrix} T^{1/2}\,\hat{\alpha} \\ T(\hat{\rho}_1-1) \\ T\,\hat{\rho}_2 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & \sigma\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr & \sigma\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr \\ \sigma\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr & \sigma^2\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & \sigma^2\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2\,dr \\ \sigma\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr & \sigma^2\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & \sigma^2\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2\,dr \end{pmatrix}$$

and $h_1 = \begin{pmatrix} \sigma W_{\underline{c}}^{\bar{c}}(1) \\ \frac{1}{2}\sigma^2\{[W_{\underline{c}}^{\bar{c}}(1)]^2-1\} \\ \frac{1}{2}\sigma^2\{[W_{\underline{c}}^{\bar{c}}(1)]^2-1\} \end{pmatrix}$.

Where in matrices if $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ is an $m\times n$ matrix with $r^{*}=rank(A)$ where B is $r^{*}\times r^{*}$ and invertible then,

$$G=\begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} \qquad (27)$$

is the generalized inverse of A, where the "0"s in equation (2) represent matrices of zeroes of dimension sufficient to make G an $n\times m$ matrix. And if an equation can be represented as $Ax=h$, $x\in R$,where the value of determinant of A equal to zero i.e. it's a non-invertible or a singular matrix then to obtain the inverse of the matrix A a generalized inverse is need to be used, x is a vector or a matrix of unknown elements, h is vector or a matrix that has the same order as the product of Ax and to obtain the forms of unknown elements of x the following equation is need to be used:

$$x = Gh + (I-GA)\,z \ ,\ z\in R \qquad (28)$$

where I is an identity matrix, z is a vector or a matrix of real numbers and G is the generalized inverse of the matrix A that satisfied $AGA = A$. Sawyer (2008)

Since the value of the determinant of A_1 is equal to zero, a generalized inverse for A_1 is need to be used. There is a generalized inverse G_{11} of A_1 will obtained by using equation (27) as follows:

$$G_{11} = \frac{1}{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \begin{pmatrix} \sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr & -\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr & 0 \\ -\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(3\times 3)}$$

Now to obtain the forms of elements of x_1 in equation (26), equation (28) will be used as follows:

Since:

$$G_{11} A_1 = = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } G_{11} h_1 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

$$\text{Where, } l_1 = \frac{[\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr]\,[\sigma\, W_{\underline{c}}^{\bar{c}}(1)] - [\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\tfrac{1}{2}\sigma^2 \{ [W_{\underline{c}}^{\bar{c}}(1)]^2 - 1\}]}{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2}$$

$$l_2 = \frac{[-\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\sigma W_{\underline{c}}^{\bar{c}}(1)] + \tfrac{1}{2}\sigma^2 \{ [W_{\underline{c}}^{\bar{c}}(1)]^2 - 1\}}{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \quad , \quad l_3 = 0$$

Then, by using equation (28) it can be concluded that:

$$x_1 = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} + \left\{ I_3 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

Where $z's$ are real numbers, then the asymptotic distributions of $T^{1/2}\hat{\alpha}$, $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$\left.\begin{aligned} &1)\, T^{1/2}\hat{\alpha} \xrightarrow{d} \frac{[\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr]\,[\sigma\, W_{\underline{c}}^{\bar{c}}(1)] - [\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\tfrac{1}{2}\sigma^2 \{ [W_{\underline{c}}^{\bar{c}}(1)]^2 - 1\}]}{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \\ &2)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} \frac{[-\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\sigma W_{\underline{c}}^{\bar{c}}(1)] + \tfrac{1}{2}\sigma^2 \{ [W_{\underline{c}}^{\bar{c}}(1)]^2 - 1\}}{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\sigma \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} - z_3 \\ &3)\, T\hat{\rho}_2 \xrightarrow{d} z_3 \,,\; z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T} \end{aligned}\right\} \quad (29)$$

Corollary (1): If there is another generalized inverse G_{12} of A_1 that can be obtained by using equation (27), it will be as follows:

$$G_{12}=\frac{1}{\sigma^2\int_0^1[W_{\varepsilon}^{c}(r)]^2dr-\left\{\sigma\int_0^1[W_{\varepsilon}^{c}(r)]\,dr\right\}^2}\begin{pmatrix}\sigma^2\int_0^1[W_{\varepsilon}^{c}(r)]^2dr & 0 & -\sigma\int_0^1[W_{\varepsilon}^{c}(r)]\,dr\\ 0 & 0 & 0\\ -\sigma\int_0^1[W_{\varepsilon}^{c}(r)]\,dr & 0 & 1\end{pmatrix}_{(3\times3)} \quad (30)$$

Then, the asymptotic distributions of $T^{1/2}\hat{\alpha}$, $T(\hat{\rho}_1-1)$ and $T\hat{\rho}_2$ can be obtained as above.

2.3 Asymptotic Distributions of the $t-type$ Statistics Under Different Tests of Hypothesis

In addition to the previous tests in (2.2), the tests that are based on $t-type$ statistics for the estimators $\hat{\alpha}, \hat{\rho}_1$ and $\hat{\rho}_2$ under the test $H_0:\alpha=0,\ \rho_1=1,\ \rho_2=0$, (i.e. $y_t=y_{t-1}+e_t$) against $H_a:\alpha\neq0$, $|\rho_1|<1$, $|\rho_2|<1$, (i.e. $y_t=\alpha+\rho_1\,y_{t-1}+\rho_2\,y_{t-2}+e_t$) will be derived as follows:

Lemma (3): If the variance-covariance matrix of the estimators of model (1) under the null hypothesis, $H_0:\alpha=0,\rho_1=1,\ \rho_2=0$ that can be written in matrix form as:

$$Var(\hat{\beta})=S_T^2\,(X'X)^{-1} \quad (31)$$

Such that,

$$\left.\begin{array}{l}
1)\,Var(\hat{\beta})=\begin{pmatrix}Var(\hat{\alpha}) & Cov(\hat{\rho}_1,\hat{\alpha}) & Cov(\hat{\rho}_2,\hat{\alpha})\\ Cov(\hat{\rho}_1,\hat{\alpha}) & Var(\hat{\rho}_1) & Cov(\hat{\rho}_1,\hat{\rho}_2)\\ Cov(\hat{\rho}_2,\hat{\alpha}) & Cov(\hat{\rho}_1,\hat{\rho}_2) & Var(\hat{\rho}_2)\end{pmatrix}\\
2)\,(X'X)^{-1}=\begin{pmatrix}T & \sum_{t=1}^{T}y_{t-1} & \sum_{t=1}^{T}y_{t-2}\\ \sum_{t=1}^{T}y_{t-1} & \sum_{t=1}^{T}y_{t-1}^2 & \sum_{t=1}^{T}y_{t-1}\,y_{t-2}\\ \sum_{t=1}^{T}y_{t-2} & \sum_{t=1}^{T}y_{t-1}\,y_{t-2} & \sum_{t=1}^{T}y_{t-2}^2\end{pmatrix}^{-1}\\
3)\,S_T^2=\sum_{t=1}^{T}(y_t-\hat{\alpha}-\hat{\rho}_1\,y_{t-1}-\hat{\rho}_2\,y_{t-2})^2/(T-3)=\sum_{t=1}^{T}\hat{e}_t^2/(T-3)
\end{array}\right\} \quad (32)$$

Then, the asymptotic distributions for $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be as follows:

$$1)\; t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\,Var(\hat{\alpha})]^{-1/2} \xrightarrow{d} l_1\, v_1^{-1/2}$$
$$2)\; t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2\,Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} (l_2 - z_3)\, v_2^{-1/2}$$
$$3)\; t_{\hat{\rho}_2} = [T\,\hat{\rho}_2][T^2\,Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_3\, v_3^{-1/2}$$
$$, z_3 \in \underline{c}\,\sqrt{T} \text{ or } \bar{c}\,\sqrt{T},\; z_{32} \in \underline{c}\,\sqrt{T},\; z_{33} \in \bar{c}\,\sqrt{T}$$

Where, l_1 and l_2 are defined as in lemma (2), $v_1 = \dfrac{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right)^2}$,

$$v_2 = \frac{1-\sigma^2 z_{32}}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right)^2} \text{ and } v_3 = \frac{\sigma^2 z_{33}}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right)^2}.$$

Proof:

By multiplying equation (31) by the scaling matrix ψ_T as follows:

$$\psi_T\, Var(\hat{\beta})\psi_T = S_T^2\,(\psi_T^{-1} X'X \psi_T^{-1})^{-1} \qquad (33)$$

Where ψ_T is defined as in lemma (2).

Then, by substituting from equations (32 (1,2)) in equation (33), the variance-covariance matrix will be:

$$\begin{pmatrix} T\,Var(\hat{\alpha}) & T^{3/2}\,Cov(\hat{\rho}_1,\hat{\alpha}) & T^{3/2}\,Cov(\hat{\rho}_2,\hat{\alpha}) \\ T^{3/2}\,Cov(\hat{\rho}_1,\hat{\alpha}) & T^2\,Var(\hat{\rho}_1) & T^2\,Cov(\hat{\rho}_1,\hat{\rho}_2) \\ T^{3/2}\,Cov(\hat{\rho}_2,\hat{\alpha}) & T^2\,Cov(\hat{\rho}_1,\hat{\rho}_2) & T^2\,Var(\hat{\rho}_2) \end{pmatrix} = S_T^2\, B_1 \qquad (34)$$

Where:

$$B_1 = \begin{pmatrix} 1 & T^{-3/2}\sum_{t=1}^T y_{t-1} & T^{-3/2}\sum_{t=1}^T y_{t-2} \\ T^{-3/2}\sum_{t=1}^T y_{t-1} & T^{-2}\sum_{t=1}^T y_{t-1}^2 & T^{-2}\sum_{t=1}^T y_{t-1}\,y_{t-2} \\ T^{-3/2}\sum_{t=1}^T y_{t-2} & T^{-2}\sum_{t=1}^T y_{t-1}\,y_{t-2} & T^{-2}\sum_{t=1}^T y_{t-2}^2 \end{pmatrix}^{-1}$$

As $T \to \infty$ and from the weak law of large number, Bell (2015), then the convergence in probability of S_T^2 is as follows:

$$S_T^2 = \sum_{t=1}^T \hat{e}_t^2 / (T-3) \xrightarrow{p} \sigma^2 \qquad (35)$$

From equations (9 (1,3)), $T^{-3/2}\sum_{t=1}^T y_{t-1}$ and $T^{-2}\sum_{t=1}^T y_{t-1}^2$ convergence in distribution to $\sigma \int_0^1 W(r)_{\underline{c}}^{\bar{c}}\,dr$ and $\sigma^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2\,dr$ respectively. Also, from equations (11 (1,3,4)), $T^{-3/2}\sum_{t=1}^T y_{t-2}$ and $T^{-2}\sum_{t=1}^T y_{t-2}^2$ ($T^{-2}\sum_{t=1}^T y_{t-1}\,y_{t-2}$) convergence in distribution to $\sigma \int_0^1 W_{\underline{c}}^{\bar{c}}(r)\,dr$ and $\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr$ respectively.

Then, as $T \to \infty$ and by using the above results equation (34) will be as follows:

$$x_2 = A_2^{-1} h_2 \ , \ x_2 \in R_{(3\times3)} \tag{36}$$

Where:

$$x_2 = \lim_{T\to\infty} \begin{pmatrix} T\,Var(\hat{\alpha}) & T^{3/2}\,Cov(\hat{\rho}_1,\hat{\alpha}) & T^{3/2}\,Cov(\hat{\rho}_2,\hat{\alpha}) \\ T^{3/2}\,Cov(\hat{\rho}_1,\hat{\alpha}) & T^2\,Var(\hat{\rho}_1) & T^2\,Cov(\hat{\rho}_1,\hat{\rho}_2) \\ T^{3/2}\,Cov(\hat{\rho}_2,\hat{\alpha}) & T^2\,Cov(\hat{\rho}_1,\hat{\rho}_2) & T^2\,Var(\hat{\rho}_2) \end{pmatrix}, h_2 = I_3 \ ,$$

$$A_2 = \begin{pmatrix} \frac{1}{\sigma^2} & \frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & \frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr \\ \frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr \\ \frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr \end{pmatrix}$$

and A_2 is the asymptotic distribution of the matrix $S_T^2 B_1$.

Since, $|A_2| = 0$ a generalized inverse G_{21} of A_2 will be obtained by using equation (27) and it will be as:

$$G_{21} = \frac{\sigma^2}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right)^2} \begin{pmatrix} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & -\frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & 0 \\ -\frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & \frac{1}{\sigma^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(3\times3)}$$

Now to obtain the forms of elements of x_2 in equation (36), equation (28) will be used, the forms of the asymptotic distributions of $T\,Var(\hat{\alpha})$, $T^2\,Var(\hat{\rho}_1)$ and $T^2\,Var(\hat{\rho}_2)$, and the asymptotic distributions for $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be derived as follows:

Since:

$$G_{21} A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$G_{21} h_2 = \frac{\sigma^2}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right)^2} \begin{pmatrix} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & -\frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & 0 \\ -\frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & \frac{1}{\sigma^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, by using equation (28) it can be concluded that:

$$x_2 = \frac{\sigma^2}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right)^2} \begin{pmatrix} \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr & -\frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr & 0 \\ -\frac{1}{\sigma}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr & \frac{1}{\sigma^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$\left\{ I_3 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}$$

Where $z's$ are real numbers, the asymptotic distributions of $T\,Var(\hat{\alpha}), T^2\,Var(\hat{\rho}_1)$ and $T^2\,Var(\hat{\rho}_2)$ will be as follows:

$$\left.\begin{aligned} &1)\, T\,Var(\hat{\alpha}) \xrightarrow{d} \frac{\sigma^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right)^2} > 0 \\ &2)\, T^2\,Var(\hat{\rho}_1) \xrightarrow{d} \frac{1-\sigma^2 z_{32}}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right)^2} > 0 \\ &3)\, T^2\,Var(\hat{\rho}_2) \xrightarrow{d} \frac{\sigma^2 z_{33}}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right)^2} > 0 \end{aligned}\right\} \quad (37)$$

To achieve the variances in equation (37) to be positive, $z_{32} \le 0$ and it is assumed to be $z_{32} \in \underline{c}\sqrt{T}$ and $z_{33} > 0$ and it is assumed to be $z_{33} \in \bar{c}\sqrt{T}$.

The $t-type$ statistics for the estimators $\hat{\alpha}$, $\hat{\rho}_1$ and $\hat{\rho}_2$ will be:

$$\left.\begin{aligned} &1)\, t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\,Var(\hat{\alpha})]^{-1/2} \\ &2)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2\,Var(\hat{\rho}_1)]^{-1/2} \\ &3)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2\,Var(\hat{\rho}_2)]^{-1/2} \end{aligned}\right\} \quad (38)$$

Then, by substituting from equation (29) that contains the asymptotic distributions of OLS estimators $T^{1/2}\hat{\alpha}, T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$, and equation (37) in equation (38), the asymptotic distributions for $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ respectively will be:

$$\left.\begin{aligned} &1)\, t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\,Var(\hat{\alpha})]^{-1/2} \xrightarrow{d} l_1\, v_1^{-1/2} \\ &2)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2\,Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} (l_2 - z_3)\, v_2^{-1/2} \\ &3)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2\,Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_3\, v_3^{-1/2} \\ &\quad , z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}, z_{32} \in \underline{c}\sqrt{T}, z_{33} \in \bar{c}\sqrt{T} \end{aligned}\right\} \quad (39)$$

Corollary (2): If there is another generalized inverse G_{22} of A_2 that can be obtained by using equation (27), it will be as follows:

$$G_{22}=\frac{\sigma^{2}}{\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2\,dr-\left(\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr\right)^2}\begin{pmatrix}\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & -\frac{1}{\sigma}\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr & 0\\ 0 & 0 & 0\\ -\frac{1}{\sigma}\int_0^1[W_{\underline{c}}^{\bar{c}}(r)]\,dr & 0 & \frac{1}{\sigma^{2}}\end{pmatrix}_{(3\times3)}$$

Then, the asymptotic distributions for $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ can be obtained as above.

3. Simulation Study

A simulation study is used to obtain bias, MSE, Thiel's U under the null hypothesis $H_0: y_t = y_{t-1} + e_t$. Also, the same measures and the power of the test under the alternative hypothesis H_a with constant term will be obtained in case of five samples size T = 30, 50, 100, 200 and 500 for five boundaries value $\bar{c}=-\underline{c}=0.3, 0.5, 0.7, 0.9$ and 1.1 by 5000 replications as follows:

OLS estimators of bounded AR (2) model with constant and with independent errors which obtained in lemma (2) that used the generalized inverse G_{11} and in corollary (1) that used the generalized inverse G_{12} are used to obtain the bias, MSE, Thiel's U and the power of the test and from table (1) in appendix the results can be summarized for five samples size T = 30, 50, 100, 200 and 500 in the following table:

Table (1)

<table>
<tr><td rowspan="2">$\bar{c}=-\underline{c}$</td><td colspan="2">G₁₁</td><td colspan="2">G₁₂</td></tr>
<tr><td>MSE</td><td>Thiel's U</td><td>MSE</td><td>Thiel's U</td></tr>
<tr><td>0.3</td><td colspan="2" rowspan="5">H_a</td><td>H_a</td><td>H_0</td></tr>
<tr><td>0.5</td><td colspan="2" rowspan="4">H_a</td></tr>
<tr><td>0.7</td></tr>
<tr><td>0.9</td></tr>
<tr><td>1.1</td></tr>
</table>

It can be notice from table (1) that both G_{11} and G_{12} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except G_{12} for value of Thiel's U and $\bar{c}=-\underline{c}=0.3$. Also, the values of the power of the test for both G_{11} and G_{12} are equal to integer one and approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$.

4. Conclusions

1. The asymptotic distributions of OLS estimators and the $t-type$ statistics of OLS estimators of bounded AR (2) model with constant and with independent errors under different tests of hypothesis have been derived.

2. For each sample size T when the values of $\bar{c}=-\underline{c}$ are increasing the values of MSE are increasing and the values of Thiel's U are decreasing for both generalized inverses G_{11} and G_{12} under both the null hypothesis H_0 and the alternative hypothesis H_a.

3. For generalized inverse G_{11} when the samples size T are increasing at the same values of $\bar{c}=-\underline{c}$, the values of MSE are decreasing for all values of $\bar{c}=-\underline{c}$ under the null hypothesis H_0, constant for $\bar{c}=-\underline{c}=0.3$ and 0.5 and constant or decreasing for $\bar{c}=-\underline{c}=0.7, 0.9$ and 1.1 under the alternative hypothesis H_a, while the values of Thiel's U are decreasing for all values of $\bar{c}=-\underline{c}$ under both the null hypothesis H_0 and the alternative hypothesis H_a.

4. For generalized inverse G_{12} when the samples size T are increasing at the same values of $\bar{c}=-\underline{c}$, the values of MSE are decreasing or constant for $\bar{c}=-\underline{c}=0.3$ and 0.5 and constant or increasing for $\bar{c}=-\underline{c}=0.7, 0.9$ and 1.1 under the null hypothesis H_0, decreasing or constant for $\bar{c}=-\underline{c}=0.3$ and 0.5, fluctuate for $\bar{c}=-\underline{c}=0.7$ and increasing for $\bar{c}=-\underline{c}=0.9$ and 1.1 under the alternative hypothesis H_a, while the values of Thiel's U are decreasing for all values of $\bar{c}=-\underline{c}$ under both the null hypothesis H_0 and the alternative hypothesis H_a.

5. For generalized inverse G_{11} for all sample sizes T and for all values of $\bar{c}=-\underline{c}$ the values of MSE and Thiel's U approved the alternative hypothesis H_a.

6. For generalized inverse G_{12} for all sample sizes T and for all values of $\bar{c}=-\underline{c}$ the values of MSE and Thiel's U approved the alternative hypothesis H_a

except for value of $\bar{c} = -\underline{c} = 0.3$ the value Thiel's U approved the null hypothesis H_0.

7. The values of the power of the test for both G_{11} and G_{12} are equal to integer one and approved the alternative hypothesis H_a for all sample sizes T and for all values of $\bar{c} = -\underline{c}$.

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Appendix of Tables

Table (1)
Bias, MSE, Thiel's U and Power of the test of bounded AR (2) model with constant and with independent errors

T	$\underline{c} = -\bar{c}$		Under H_0			Under H_a					
			bias ρ_1	MSE	Thiel's U	bias α	bias ρ_1	bias ρ_2	MSE	Thiel's U	Power
30	0.3	G_{11}	-0.205	0.046	0.230	-0.086	-0.157	0.025	0.030	0.182	1
		G_{12}	0.055	0.032	0.180	-0.086	0.105	-0.237	0.031	0.181	1
	0.5	G_{11}	-0.229	0.147	0.236	-0.087	-0.182	0.061	0.083	0.178	1
		G_{12}	0.091	0.097	0.177	-0.087	0.141	-0.262	0.085	0.174	1
	0.7	G_{11}	-0.244	0.355	0.231	-0.087	-0.196	0.098	0.167	0.163	1
		G_{12}	0.128	0.217	0.166	-0.087	0.178	-0.276	0.161	0.156	1
	0.9	G_{11}	-0.259	0.758	0.226	-0.087	-0.210	0.134	0.282	0.147	1
		G_{12}	0.164	0.437	0.155	-0.087	0.214	-0.290	0.254	0.136	1
	1.1	G_{11}	-0.274	1.589	0.225	-0.094	-0.226	0.171	0.429	0.131	1
		G_{12}	0.201	0.871	0.147	-0.094	0.251	-0.306	0.358	0.116	1
50	0.3	G_{11}	-0.137	0.042	0.168	-0.091	-0.089	0.012	0.030	0.143	1
		G_{12}	0.042	0.032	0.141	-0.091	0.092	-0.169	0.030	0.143	1
	0.5	G_{11}	-0.154	0.134	0.168	-0.090	-0.106	0.041	0.083	0.136	1
		G_{12}	0.071	0.096	0.137	-0.090	0.121	-0.186	0.083	0.135	1
	0.7	G_{11}	-0.170	0.325	0.163	-0.092	-0.121	0.069	0.165	0.124	1
		G_{12}	0.099	0.218	0.128	-0.092	0.149	-0.201	0.158	0.120	1
	0.9	G_{11}	-0.184	0.714	0.160	-0.097	-0.135	0.097	0.278	0.112	1
		G_{12}	0.127	0.453	0.121	-0.097	0.177	-0.215	0.252	0.105	1
	1.1	G_{11}	-0.200	1.518	0.158	-0.091	-0.150	0.126	0.422	0.098	1
		G_{12}	0.156	0.930	0.115	-0.091	0.206	-0.230	0.359	0.089	1
100	0.3	G_{11}	-0.079	0.038	0.113	-0.096	-0.030	0.000	0.030	0.103	1
		G_{12}	0.030	0.032	0.102	-0.096	0.080	-0.110	0.030	0.103	1
	0.5	G_{11}	-0.093	0.121	0.111	-0.097	-0.044	0.020	0.083	0.097	1
		G_{12}	0.050	0.096	0.098	-0.097	0.100	-0.124	0.082	0.096	1
	0.7	G_{11}	-0.106	0.292	0.107	-0.097	-0.056	0.040	0.164	0.087	1
		G_{12}	0.070	0.219	0.091	-0.097	0.120	-0.136	0.158	0.085	1
	0.9	G_{11}	-0.118	0.645	0.103	-0.097	-0.068	0.060	0.274	0.077	1
		G_{12}	0.090	0.465	0.085	-0.097	0.140	-0.148	0.255	0.073	1
	1.1	G_{11}	-0.131	1.396	0.102	-0.098	-0.081	0.080	0.416	0.068	1
		G_{12}	0.110	0.987	0.083	-0.098	0.160	-0.161	0.368	0.063	1
200	0.3	G_{11}	-0.046	0.035	0.077	-0.097	0.003	-0.009	0.030	0.073	1
		G_{12}	0.021	0.032	0.072	-0.097	0.071	-0.077	0.030	0.073	1
	0.5	G_{11}	-0.058	0.113	0.076	-0.099	-0.008	0.005	0.083	0.068	1
		G_{12}	0.035	0.096	0.069	-0.099	0.085	-0.088	0.082	0.068	1
	0.7	G_{11}	-0.067	0.266	0.073	-0.100	-0.018	0.019	0.163	0.062	1
		G_{12}	0.049	0.219	0.066	-0.100	0.099	-0.098	0.159	0.062	1
	0.9	G_{11}	-0.077	0.589	0.070	-0.097	-0.028	0.034	0.272	0.054	1
		G_{12}	0.064	0.469	0.062	-0.097	0.114	-0.108	0.258	0.053	1
	1.1	G_{11}	-0.088	1.350	0.068	-0.101	-0.038	0.048	0.410	0.046	1
		G_{12}	0.078	1.045	0.058	-0.101	0.128	-0.118	0.375	0.044	1
500	0.3	G_{11}	-0.023	0.034	0.047	-0.099	0.027	-0.017	0.030	0.045	1
		G_{12}	0.013	0.032	0.045	-0.099	0.063	-0.053	0.030	0.045	1
	0.5	G_{11}	-0.031	0.106	0.046	-0.098	0.019	-0.008	0.083	0.043	1
		G_{12}	0.022	0.096	0.044	-0.098	0.072	-0.061	0.083	0.043	1
	0.7	G_{11}	-0.038	0.248	0.044	-0.101	0.012	0.001	0.163	0.039	1
		G_{12}	0.031	0.220	0.042	-0.101	0.081	-0.068	0.160	0.039	1
	0.9	G_{11}	-0.046	0.551	0.042	-0.099	0.004	0.010	0.270	0.034	1
		G_{12}	0.040	0.476	0.039	-0.099	0.090	-0.076	0.261	0.033	1
	1.1	G_{11}	-0.053	1.257	0.041	-0.098	-0.003	0.019	0.406	0.028	1
		G_{12}	0.049	1.071	0.037	-0.098	0.099	-0.083	0.383	0.028	1